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Completion of band-limited data sets on the sphere

W.-D. Schuh, S. Müller and J.M. Brockmann

Abstract In this study we propose the complementation of satellite-only gravity field models by additional a priori information to obtain a *complete model*. While the accepted gravity field models are restricted to a sub-domain of the frequency space, the complete models form a complete basis in the entire space, which can be represented in the frequency domain (spherical harmonics) as well as in the space domain (data grids). The additional information is obtained by the smoothness of the potential field. Using this a priori knowledge, a stochastic process on the sphere is established as a background model. The measurements of satellite-only models are assimilated to this background model by a subdivision into the commission, transition and omission sub-domain. Complete models can be used for a rigorous fusion of complementary data sets in a multi-mission approach and guarantee also, as stand-alone gravity-field models, full-rank variance/covariance matrices for all vector-valued, linearly independent functionals.

Keywords complete models, stochastic processes on the sphere, spherical harmonics, covariance functions, smoothness of potential fields, variance/covariance estimation

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1 Introduction

The observation of the Earth's system is one of the most important research topics of these days. Although huge sets of data are already available, additional measurements are still necessary to obtain a better understanding of the Earth's processes. Measurements from satellites have the great advantage of delivering homogeneous data sets over large areas (oceans or continents) or covering the whole globe. Due to special measurement conditions with respect to the satellite orbits as well as the observation technique, such data sets often have a band-limited spatial or spectral resolution. For example, gravity field model have, due to the downward continuation process, only a restricted frequency resolution. The global behavior of long-wavelength information is well determined, whereas the short-wavelength content is only weakly determined or not estimable. To avoid unstable systems gravity field models are often restricted to a special sub-domain, e.g. spherical harmonics up to a maximal degree. Users of such models must keep in mind that these models only describe a part (commission domain) of the real phenomena and that also the omission domain has to be taken into account. This fact is well known and was often discussed (cf. e.g. Losch et al., 2002). One typical example is the determination of the mean dynamic ocean topography as the difference between the mean sea surface height and the geoid height (Becker et al., 2011). The trace-wise spatially highly resolved altimetric measurements have to be combined with the spectral band-limited information of the gravity field and computed on a predefined grid of an ocean circulation model. Beside the different representations of the data, especially the different information content is crucial for the modeling. To overcome the shortcomings of band-limitation and incomplete information representation two approaches are possible: A restriction of the model to a least common sub-domain or an extension to the entire space by additional a priori information.

Special filter processes are usually introduced to homogenize all the available information with respect to a least common subspace (Jekeli, 1981, 1996; Wahr et al., 1998; Swenson and Wahr, 2006). It goes without saying that with this filter approach also valuable information of the signal is filtered out. To avoid this drawback of the filter processes, a rigorous fusion of the gravity field and a priori smoothness conditions is here proposed to form a complete model. Kusche (2007) showed an appropriate way to adapted smoothness conditions regarding preprocessed unconstrained gravity field models. In contrast to Kusche (2007), we extend our model to the complete space including the commission as well as the omission space.

The concept of our approach is characterized by the following items. As background model we introduce the Hilbert space H^1_Γ, as a complete and separable space for continuous functions on the sphere Γ, with square integrable first derivatives. We use an isotropic stationary stochastic process on the sphere to represent this background model. This process can be represented in form of random coefficients as well as in form of a covariance function. This flexibility in representing the stochastic process allows one now to treat this information individually for the commission and omission space. The spherical harmonics as base functions allow us, due to the orthogonality relations, to split up the Hilbert space into sub-domains. With respect to the gravity field models we divide the space into three subdomains: commission, transition and omission domain. The commission sub-domain is mainly fixed by the real measurements (e.g. satellite-to satellite tracking data, gravity gradient measurements, ...). In the transition zone the information of measurements is supported by the stochastic background process in form of random coefficients, which are modeled by their expectations and variances. This additional a priori information prevents the well-known over-estimation of information content for high frequencies. And finally, the omission domain up to infinity is dominated only by the a priori knowledge about the smoothness of the potential field modeled in form of covariance functions. The shape of the covariance function is given by the theoretical assumptions on the smoothness or by other a priori information. For degrees up to 360, 720 or 2160 the knowledge about the smoothness can be supported by high-resolution gravity field models (e.g. EGM96, EGM08), but up to infinity only theoretical assumptions, like smoothness conditions with respect to the Hilbert space H^1_Γ can be used (Meissl, 1971; Schuh and Becker, 2010). .

The main point is now that we construct with this approach a complete model which enables us to express any functionals and also their variance/covariance information in a consistent way. All functionals of this model are unbiased estimable functionals and can therefore be used without filter processes for data assimilation as well as for hypothesis testing. Because of the completeness with respect to the frequency domain, all vector-valued functions with linear independent functionals (e.g. profiles, gridded data,...) possess an invertible variance/covariance matrix.

This paper is organized in the following way. The construction of stochastic processes on the sphere is discussed in Section 2 before we introduce the model building process and the separation into sub-domains in Section 3. An example with a GRACE and combined GRACE/GOCE field provides a proof of concept in the final section. A summary concludes this article.

2 Stochastic processes on the sphere

Consider an arbitrary square integrable function $u(\vartheta,\lambda)$ on the unit sphere Γ. Because of the completeness of the orthonormalized Laplace's surface spherical harmonics $\bar{C}_{\ell m}(\vartheta,\lambda)$ and $\bar{S}_{\ell m}(\vartheta,\lambda)$ this function can be written in form of a spherical harmonic synthesis

$$u(\vartheta,\lambda)=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[\bar{c}_{\ell m}\bar{C}_{\ell m}(\vartheta,\lambda)+\bar{s}_{\ell m}\bar{S}_{\ell m}(\vartheta,\lambda)\right] . \tag{1}$$

Treating the spherical harmonic coefficients $\bar{c}_{\ell m}$ and $\bar{s}_{\ell m}$ as random variables $\bar{\mathcal{C}}_{\ell m}$ and $\bar{\mathcal{S}}_{\ell m}$ the function $u(\vartheta,\lambda)$ becomes a random process $\mathcal{U}(\vartheta,\lambda)$ on the sphere,

$$\mathcal{U}(\vartheta,\lambda)=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[\bar{\mathcal{C}}_{\ell m}\bar{C}_{\ell m}(\vartheta,\lambda)+\bar{\mathcal{S}}_{\ell m}\bar{S}_{\ell m}(\vartheta,\lambda)\right] . \tag{2}$$

We will assume now that the random variables have zero expectations, i.e.

$$E\left\{\bar{\mathcal{C}}_{\ell m}\right\} = E\left\{\bar{\mathcal{S}}_{\ell m}\right\} = 0 \qquad \ell = 0,\ldots,\infty, m = 0,\ldots,\ell \tag{3}$$

are mutually uncorrelated, i.e.

$$\left.\begin{aligned}\Sigma\left\{\bar{\mathcal{C}}_{\ell m},\bar{\mathcal{C}}_{sr}\right\} &= 0\\ \Sigma\left\{\bar{\mathcal{S}}_{\ell m},\bar{\mathcal{S}}_{sr}\right\} &= 0\end{aligned}\right\} \text{ if s} \neq \ell \text{ or r} \neq \text{m or both}$$
$$\Sigma\left\{\bar{\mathcal{C}}_{\ell m},\bar{\mathcal{S}}_{sr}\right\} = 0 \qquad \text{in any case} \tag{4}$$

and have constant variances per degree, i.e.

$$\Sigma\left\{\bar{\mathcal{C}}_{\ell m}\right\} = \Sigma\left\{\bar{\mathcal{S}}_{\ell m}\right\} := \frac{1}{2\ell+1}\Sigma\left\{\mathcal{S}_\ell\right\} \tag{5}$$

where $\Sigma\left\{\mathcal{S}_\ell\right\} = E\left\{\mathcal{S}_\ell^2\right\}$ with the random variable

$$\mathcal{S}_\ell^2 = \bar{\mathcal{C}}_{\ell 0}^2 + \sum_{m=1}^{\ell} \bar{\mathcal{C}}_{\ell m}^2 + \bar{\mathcal{S}}_{\ell m}^2 \ . \tag{6}$$

The stochastic process (2) can be characterized by the expectation $E\{\mathcal{U}(\vartheta,\lambda)\} = 0$ and the stationary, isotropic covariance function

$$\mathrm{Cov}\left\{\mathcal{U}(\vartheta,\lambda);\mathcal{U}(\vartheta',\lambda')\right\} = \sum_{\ell=0}^{\infty}\Sigma\left\{\mathcal{S}_\ell\right\}P_\ell(\cos\psi) \ , \tag{7}$$

which depends only on Legendre polynomials P_ℓ as functions of the spherical distance ψ between (ϑ,λ) and (ϑ',λ') (cf. e.g. Meissl (1971), Sect. 4; Moritz (1980), Sect. 34). In many applications the estimated degree variances

$$\bar{\sigma}_l^2 = \bar{c}_{\ell 0}^2 + \sum_{m=1}^{\ell} \bar{c}_{\ell m}^2 + \bar{s}_{\ell m}^2 \tag{8}$$

as a single realization of the stochastic variable $\mathcal{S}_\ell^2$ is used to fix $E\left\{\mathcal{S}_\ell^2\right\} = \Sigma\left\{\mathcal{S}_\ell\right\}$, or $\Sigma\left\{\mathcal{S}_\ell\right\}$ is substituted by deterministic quantities.

In the following it will be shown that a stochastic process on the sphere can also be formulated for deterministically defined degree variances by utilizing the amplitude and phase formulation. Recalling the relations between the Laplace's surface spherical harmonics $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$

$$\begin{aligned}\bar{C}_{\ell m}(\vartheta,\lambda) &= \bar{P}_{\ell m}(\cos\vartheta)\cos m\lambda\\ \bar{S}_{\ell m}(\vartheta,\lambda) &= \bar{P}_{\ell m}(\cos\vartheta)\sin m\lambda\end{aligned} \tag{9}$$

with the fully normalized Legendre functions $\bar{P}_{\ell m}$ we can express (1) using the amplitude $A_{\ell m}$ and phase $\Phi_{\ell m}$ notation

$$u(\vartheta,\lambda) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} A_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)\cos(m\lambda + \Phi_{\ell m}) \ . \tag{10}$$

The connections between the parameters $\bar{c}_{\ell m}$, $\bar{s}_{\ell m}$ and $A_{\ell m}$, $\Phi_{\ell m}$ are given by

$$\left.\begin{aligned}\bar{c}_{\ell m} &= A_{\ell m}\cos\Phi_{\ell m}\\ \bar{s}_{\ell m} &= -A_{\ell m}\sin\Phi_{\ell m}\end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned}A_{\ell m} &= \sqrt{\bar{c}_{\ell m}^2 + \bar{s}_{\ell m}^2}\\ \Phi_{\ell m} &= \tan^{-1}\tfrac{-\bar{s}_{\ell m}}{\bar{c}_{\ell m}} \ .\end{aligned}\right. \tag{11}$$

We generate now a stochastic process on the sphere by introducing the following conditions. The amplitudes $A_{\ell m}$ are constant quantities depending on the degree variances σ_ℓ^2 of degree ℓ

$$A_{\ell m}^2 = \frac{2}{2\ell+1}\,\sigma_\ell^2 \ . \tag{12}$$

The phases $\Phi_{\ell m}$ are mapped into a vector $\boldsymbol{\Phi}$ by an appropriate numbering scheme. The random counterpart is defined by $\mathcal{P}_{\ell m}$ and $\boldsymbol{\mathcal{P}}$, respectively, where the probability density function $f_{\boldsymbol{\mathcal{P}}}(\mathbf{p})$ is defined by

$$\begin{aligned}f_{\boldsymbol{\mathcal{P}}}(\mathbf{p}) &= \prod_{\ell=0}^{\infty}\prod_{m=0}^{\ell} f_{\mathcal{P}_{\ell m}}(p_{\ell m})\\ f_{\mathcal{P}_{\ell m}}(p_{\ell m}) &= \begin{cases}0 & p_{\ell m} \leq -\pi\\ \frac{1}{2\pi} & -\pi \leq p_{\ell m} \leq \pi\\ 0 & \pi \leq p_{\ell m} \ .\end{cases}\end{aligned} \tag{13}$$

This means we have a uniformly distributed phase, for each degree ℓ and order m, and phases for different degrees/orders are mutually independent. The stochastic process $\mathcal{U}(\vartheta,\lambda)$ on the sphere is defined by

$$\mathcal{U}(\vartheta,\lambda) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} A_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)\cos(m\lambda + \mathcal{P}_{\ell m}) \ . \tag{14}$$

The expectation

$$E\{\mathcal{U}(\vartheta,\lambda)\} = \int_{-\infty}^{\infty} u(\vartheta,\lambda) f_{\boldsymbol{\mathcal{P}}}(\mathbf{p})d\mathbf{P} \tag{15}$$

is fixed again by $E\{\mathcal{U}(\vartheta,\lambda)\} = 0$, and the covariance

$$\begin{aligned}\operatorname{Cov}\left\{\mathcal{U}(\vartheta,\lambda),\mathcal{U}(\vartheta',\lambda')\right\} &= \\ &= \int_{-\infty}^{\infty} u(\vartheta,\lambda)\, u(\vartheta',\lambda')\, f_{\mathcal{P}}(\mathbf{p}) d\mathbf{P} \qquad (16)\end{aligned}$$

of this process, can be expressed again by the isotropic covariance function

$$\begin{aligned}\operatorname{Cov}\left\{\mathcal{U}(\vartheta,\lambda),\mathcal{U}(\vartheta',\lambda')\right\} &= \sum_{\ell=0}^{\infty} \sigma_\ell^2\, P_\ell(\cos\psi) \\ &=: \operatorname{cov}(\psi,\sigma_\ell^2) \qquad (17)\end{aligned}$$

and depends only on the degree variances σ_ℓ^2 and the spherical distance ψ (cf. appendix for a detailed derivation).

We see that the definition of a uniformly distributed phase, independent for each degree and order, yields an isotropic stochastic process. However, now the degree variances are not represented as second moments of a stochastic variable, in contrast to the first stochastic process (7). The degree variances are defined by the amplitudes, are deterministic quantities, and can be fixed arbitrarily. This allows us to distinguish rigorously between stochastic quantities, realizations and deterministic quantities. As shown in Schuh and Becker (2010), Table 1, the smoothness conditions for the Hilbert spaces H_Γ^p are formulated as asymptotic restrictions $\sigma_\ell^2 < \frac{c}{\ell^{2p-1}}$ on the degree variances. This deterministic approach coincides with the amplitude/phase formulation (10) of the stochastic process.

Nevertheless, both representations of the stochastic process in the spectral domain result in a stationary isotropic covariance function. This allows for a representation of the processes in the spectral domain by (2) and (14) as well as in the space domain by (7) and (17). Due to the orthogonality relations of the spherical harmonics (Moritz, 1980, p. 21 (3-16)) the infinite Hilbert space can be separated into subspaces, and in each subspace we can express the stochastic process in an individually appropriate way.

3 Model building and separability

The Hilbert space H_Γ^1 is separated into different subspaces. We divide the space H_Γ^1 into the commission space $(H^{(c)})_\Gamma^1$, a transition space $(H^{(t)})_\Gamma^1$, and into the omission space $(H^{(o)})_\Gamma^1$. While the content of the commission space is dominated by the measurements, the content of the omission space reflects basically the weak knowledge of the a priori model.

In the transition space the observations are supported by the a priori model, represented by a stochastic process in terms of randomized spherical harmonic coefficients as documented in (2). The degree variances of the corresponding covariance function (7) are introduced as random variables defined by the spherical harmonic coefficients with $E\left\{\bar{\mathcal{C}}_{\ell m}\right\} = E\left\{\bar{\mathcal{S}}_{\ell m}\right\} = 0$ and the variances

$$\Sigma\left\{\bar{\mathcal{C}}_{\ell m}\right\} = \Sigma\left\{\bar{\mathcal{S}}_{\ell m}\right\} = \frac{10^{-10}}{\ell^4}, \quad \begin{array}{l}\ell_{t_{min}},\ldots,\ell_{t_{max}} \\ m=0,\ldots,\ell\end{array} \qquad (18)$$

according to Kaula's rule (Kaula, 1966, p. 98 (5.51))

$$\Sigma\left\{\bar{\mathcal{S}}_\ell\right\} = \frac{10^{-10}(2l+1)}{\ell^4} \qquad (19)$$

considering again the relation (5). $\ell_{t_{min}}$ and $\ell_{t_{max}}$ represent the degree range of the transition zone.

In contrast to random coefficients as prior information in the transition sub-domain, the prior information in the omission sub-domain is defined by the covariance function (17) of the stochastic process. The degree variances are represented as constant quantities. The sizes of these quantities are fixed with respect to Kaula's rule

$$\sigma_\ell^2 = \frac{10^{-10}(2\ell+1)}{\ell^4}, \qquad \ell = \ell_{o_{min}},\ldots,\ell_{o_{max}} \qquad (20)$$

where $\ell_{o_{min}}$ and $\ell_{o_{max}}$ defines the range of the omission sub-domain.[1]

[1] For practical reasons we work with $\ell_{o_{max}} = 18,000$. It is well known that the variance $\operatorname{cov}(0,\sigma_1^2)$ of the stochastic process using Kaula's degree variances (20) is finite and given by

$$\begin{aligned}\operatorname{cov}(0,\sigma_\ell^2) &= 10^{-10}\sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell^4} \\ &= 10^{-10}\left(2\sum_{\ell=1}^{\infty}\frac{1}{l^3}+\sum_{\ell=1}^{\infty}\frac{1}{l^4}\right) \\ &= 10^{-10}\left(2\zeta(3)+\zeta(4)\right), \qquad (21)\end{aligned}$$

where $\zeta(3)$ and $\zeta(4)$ denote the function values of Riemann's zeta function. R. Apéry proved in 1977 that $\zeta(3)$ is irrational with a value of $\zeta(3) = 1.20205690315959\ldots$ (Hata, 2000). Euler (1740) p.133, § 18 already derived $\zeta(4) = \frac{\pi^4}{90}$. These constants can be used to compute the relative approximation error for the finite summation up to 18,000 with $1\cdot 10^{-4}$ $(2\cdot 10^{-4})$ starting the omission space at 181 (251).

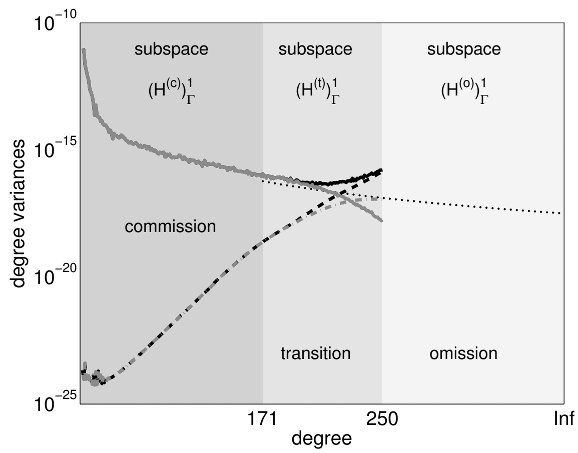

Fig. 1 Complete model: commission - transition - omission subspaces. The solid lines mark the degree variances of the signal in the combined *GRACE/GOCE* model (black) in contrast to the *complete model* (gray). The dashed lines reflect the accuracy of the signals. The dotted line marks Kaula's rule for the transition and omission sub-domain.

Due to the orthogonality relations (Moritz, 1980, p. 21 (3-16)) of the spherical harmonics, the subspaces are orthogonal to each other for continuous sampling. In our case the discrete satellite measurements have a very uniform distribution of data points over the sphere and therefore our subspaces are almost orthogonal, as will be seen later on.

4 Application and Simulations

To study the performance of complete models with state-of-the-art gravity field models we assimilate the normal equations of two gravity field models with the a priori model: the static solution of the *ITG-Grace2010s* model (Mayer-Gürr et al., 2010) and a combined *GRACE/GOCE* model including 7 months of GOCE measurements (Pail et al., 2011). Many tests show that the exact definition of the ranges for the commission, transition and omission domain is not crucial. For the computation we used the following partitioning strategies. In the case of the *ITG-Grace2010s* the transition sub-domain is defined in the range between degree 151 up to 180, and in case of the combined *GRACE/GOCE* model we choose 171 up to 250.

To assess the behavior of the different models we implemented an error propagation into height anomalies with the variance/covariance information of the particular gravity field models. Tables 1 and 2 summarize the characteristic values. For the *ITG-Grace2010s* (*GRACE/GOCE*) model the standard deviation of the complete model of $\pm 41.7cm$ ($\pm 42.7cm$) can be split up into the deviations $\pm 2.9cm$ ($\pm 2.6cm$) in the commission zone, $\pm 22.0cm$ ($\pm 34.2cm$) in the transition zone and $\pm 35.5cm$ ($\pm 25.5cm$) in the omission zone. As a first point we can see that the introduction of a priori information in the transition zone reduces the standard deviation of the height anomalies to $\pm 37.5cm$ ($\pm 30.1cm$). We can state that the standard deviation up to 180 (250) is reduced by the a priori information from $\pm 22cm$ to $\pm 12cm$ ($\pm 34cm$ to $\pm 16cm$), but the additional omission error of $\pm 35cm$ ($\pm 25cm$) yields again a very inferior overall performance. Note also that the overall standard deviation of the combined *GRACE/GOCE* model is larger than that of the *ITG-Grace2010s* before adding a priori information. This is a clear indication for the over-parameterization of high frequencies in the transition zone for the *GRACE/GOCE* model if no prior information is introduced. This is also reflected by Fig. 1 by the increasing degree variances in the transition zone. The mixed term defined by the norm of the covariances between commission and transition space is very small. As mentioned above, these sub-domains are almost orthogonal due to the regular data distribution on the sphere within the gravity field models.

As expected the point values perform very poorly (cf. first row in Table 2 with area-size 0.000) because the request of a point value cannot be answered satisfyingly by the band-limited gravity field information. But if we ask for the standard deviation of averaged height anomalies in a quadratic region we may expect accurate values. For this experiment we compute point values on a $0.1°$ grid with variances and covariances, and compute the moving average over a square area with side lengths of $.5°$, $1°$, $2°$ and $4°$. If we choose such a fine grid the information is highly correlated. This is well documented by the shape of the corresponding covariance function (cf. Fig. 2). For the combined *GRACE/GOCE* model roughly up to degree 3 strong correlations appear. The omission part shows a smaller correlation length than the commission part. Of particular interest for the error behavior are the positive and negative slopes. In positive areas an averaging process

ITG-Grace2010s	commission (2 – 150)	transition (151 – 180)	‖mixed‖	model (2 – 180)	omission (181 – ∞)	all (2 – ∞)
without a priori information						
std. dev. [m]	0.029	0.220	0.009	0.221	0.353	0.417
with a priori information in the range from 151-180						
std. dev. [m]	0.026	0.121	0.007	0.124	0.353	0.375
GRACE/GOCE	commission (2 – 170)	transition (171 – 250)	‖mixed‖	model (2 – 250)	omission (251 – ∞)	all (2 – ∞)
without a priori information						
std. dev. [m]	0.026	0.342	0.011	0.343	0.255	0.427
with a priori information in the range from 171-250						
std. dev. [m]	0.026	0.158	0.007	0.160	0.255	0.301

Table 1 Characteristics of the accuracies of height anomalies for a point value of a complete model with commission subspace 2-150 (2-170), transition space 151-180 (171-250) and omission space. The standard deviations (std.d̃ev.) result from an error propagation using the model *ITG-Grace2010s* and the combined *GRACE/GOCE* model respectively. All values are averaged over the region $-5°$ to $5°$ latitude and $-5°$ and $5°$ longitude with a grid spacing of $0.1°$, but to be precise, these values reflect the mean behavior of the point information and not mean values over a specific region.

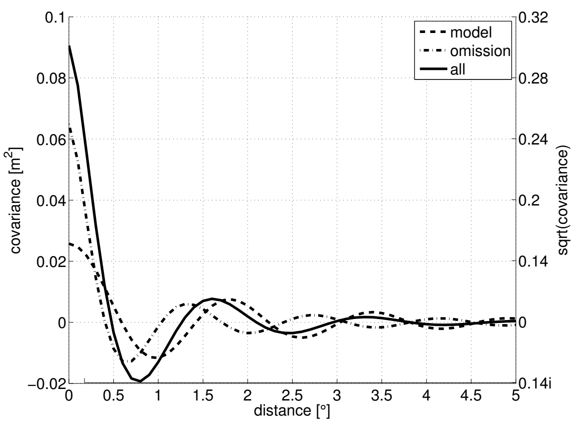

Fig. 2 Covariance information for height anomalies from the combined *GRACE/GOCE* model computed on a profile at the equator ($\phi = 0°$, $\lambda = 0° ... 5°$). The model is structured in a commission space $(2-170)$, a transition space $(171-250)$ and an omission space $(251-\infty)$. The contributions of the commission and transition space is summarized by the term 'model'. To better understand the magnitude of the variances this figure is equipped with two y-axes. The left one reflects the variances and on the right one the square root values of the left scalar are displayed.

does not gain high accuracy, because the positive correlations counteract. By contrast, the negative parts of the covariance function accelerate the benefits of an averaging process, and the accuracy decreases disproportionately. The numbers in Table 2 reflect exactly this behavior and give a comparison of the resolutions and expected accuracies for the two models.

	ITG-Grace2010s			GRACE/GOCE		
side length of the area	model (2-180)	omission (181-∞)	all (2-∞)	model (2-250)	omission (251-∞)	all (2-∞)
0.000	0.124	0.353	0.375	0.160	0.255	0.301
0.500	0.112	0.265	0.288	0.137	0.157	0.208
1.000	0.076	0.123	0.145	0.069	0.040	0.079
2.000	0.011	0.031	0.033	0.019	0.019	0.027
4.000	0.007	0.013	0.015	0.009	0.005	0.010

Table 2 Characteristics of the accuracy of the geoid undulations for mean values in a specific square area.

5 Summary and conclusions

In this article the construction and advantages of *complete models* are illustrated. A background stochastic process forms the basis for a complete model. A priori information defines the amplitude and smoothness of this background field in terms of degree variances. This background model forms a complete base which means that each square integrable function is a member of this infinite space. The stochastic process can be described equivalently in the space domain by covariance functions and in the frequency domain by spherical harmonic coefficients. Into this background model the band-limited data sets from satellite missions are assimilated in a sequential least squares approach. The resulting model is complete again and reflects exactly the strengths and weaknesses of the involved information. In addition it has the following properties:

- The variance/covariance matrices of all vector-valued functions with linearly independent functionals (e.g. for geoid undulations on a grid) have full rank, i.e. their inverses exist. This allows for a rigorous assimilation of gravity information into other Earth system models with an arbitrary grid (e.g. ocean circulation models).
- All linear functionals are unbiased estimable functions, because the complete model spans the entire space and no nullspace is left. No additional computations are necessary to prove that the linear functional is a member of the commission subdomain. This enables us to define arbitrary problem dependent functions in the space domain and compute the variance/covariance information rigorously (e.g. the mass variation over Greenland).

Complete models are universally applicable as standalone models as well as for assimilation purposes.

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Appendix

Expectation and variance of a stochastic process in amplitude/phase notation on the sphere

The stochastic process $\mathcal{U}(\vartheta,\lambda)$ on the sphere is defined by

$$\mathcal{U}(\vartheta,\lambda) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} A_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)\cos(m\lambda+\mathcal{P}_{\ell m}) \tag{22}$$

where the phases constitute random variables. The distribution is defined by

$$f_{\boldsymbol{\mathcal{P}}}(\mathbf{p}) = \prod_{\ell=0}^{\infty}\prod_{m=0}^{\ell} f_{\mathcal{P}_{\ell m}}(p_{\ell m})$$

$$\text{with} \quad f_{\mathcal{P}_{\ell m}}(p_{\ell m}) = \begin{cases} 0 & p_{\ell m} \le -\pi \\ \frac{1}{2\pi} & -\pi \le p_{\ell m} \le \pi \\ 0 & \pi \le p_{\ell m} \end{cases} . \tag{23}$$

This means we have uniformly distributed phases, for each degree ℓ and order m, and phases for different degrees/orders are mutually independent.

We are interested in the expectation

$$E\{\mathcal{U}(\vartheta,\lambda)\} = \int_{-\infty}^{\infty} u(\vartheta,\lambda) f_{\boldsymbol{\mathcal{P}}}(\mathbf{p}) d\mathbf{P}$$
$$= \int_{-\infty}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} A_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)\cos(m\lambda+\mathcal{P}_{\ell m})\, f_{\boldsymbol{\mathcal{P}}}(\mathbf{p})\, d\mathbf{P} .$$

Due to the independence of the phases the integral can be rewritten as

$$\int_{-\infty}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} A_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)\cos(m\lambda+\mathcal{P}_{\ell m})\, f_{\mathcal{P}_{\ell m}}(p_{\ell m})\ \, dp_{\ell m}.$$

If we now interchange integration and summation and introduce the individual distribution (23) we get

$$E\{\mathcal{U}(\vartheta,\lambda)\} = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} A_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)$$
$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(m\lambda+p_{\ell m})\, dp_{\ell m}$$
$$= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} A_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)\ \frac{1}{2\pi}\big(-\sin m\lambda+\sin m\lambda\big) .$$

Finally we see that

$$E\{\mathcal{U}(\vartheta,\lambda)\} = 0 . \tag{24}$$

The variance of this process is given by

$$\mathrm{Cov}\left\{\mathcal{U}(\vartheta,\lambda),\mathcal{U}(\vartheta',\lambda')\right\} =$$
$$= \int_{-\infty}^{\infty} u(\vartheta,\lambda)\, u(\vartheta',\lambda')\, f_{\boldsymbol{\mathcal{P}}}(\mathbf{p}) d\mathbf{P} . \tag{25}$$

Because of the independence of the random variables this can be written as

$$\mathrm{Cov}\left\{\mathcal{U}(\vartheta,\lambda),\mathcal{U}(\vartheta',\lambda')\right\} =$$
$$= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} A_{\ell m}^2\bar{P}_{\ell m}(\cos\vartheta)\bar{P}_{\ell m}(\cos\vartheta')$$
$$\times \int_{-\infty}^{\infty}\cos(m\lambda+p_{\ell m})\cos\left(m\lambda'+p_{\ell m}\right) f_{\mathcal{P}_{\ell m}}(p_{\ell m})\, dp_{\ell m}. \tag{26}$$

If we extend the first cosine term in the integral to $(m(\lambda-\lambda')+m\lambda'+p_{\ell m})$, use the relation

$$cos(x+y)\cos(x-y) = \frac{1}{2}(\cos 2x+\cos 2y)$$

and substitute

$$x = \frac{1}{2}m(\lambda-\lambda')+m\lambda'+p_{\ell m}$$
$$y = \frac{1}{2}m(\lambda-\lambda')$$

the integral can be solved and yields

$$\frac{1}{2}\sin\left(m(\lambda-\lambda')+2m\lambda'+2p_{\ell m}\right)\Big|_{p_{\ell m}=-\pi}^{p_{\ell m}=\pi}+ \\ +\frac{1}{2}\cos\left(m(\lambda-\lambda')\right)p_{\ell m}\Big|_{p_{\ell m}=-\pi}^{p_{\ell m}=\pi} \quad .$$

The first term vanishes because of of the skew symmetry of the sine and only the cosine term is relevant. Substituting this result into (26) yields

$$\mathrm{Cov}\left\{\mathcal{U}(\vartheta,\lambda),\mathcal{U}(\vartheta',\lambda')\right\} \\ =\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\frac{1}{2}A_{\ell m}^2\bar{P}_{\ell m}(\cos\vartheta)\bar{P}_{\ell m}(\cos\vartheta')\cos\left(m\lambda-m\lambda'\right) \quad .$$

Applying the addition theorem

$$\cos\left(m\lambda-m\lambda'\right)=\cos m\lambda\cos m\lambda'+\sin m\lambda\sin m\lambda'$$

and recalling the definition of Laplace's surface spherical harmonics (9) the right hand side can be reformulated as

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\frac{1}{2}A_{\ell m}^2\Big(\bar{C}_{\ell m}(\vartheta,\lambda)\bar{C}_{\ell m}(\vartheta',\lambda')+\bar{S}_{\ell m}(\vartheta,\lambda)\bar{S}_{\ell m}(\vartheta',\lambda')\Big) \quad .$$

Introducing now the amplitudes defined in (12) we get

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\frac{1}{2\ell+1}\sigma_\ell^2\Big(\bar{C}_{\ell m}(\vartheta,\lambda)\bar{C}_{\ell m}(\vartheta',\lambda')+\bar{S}_{\ell m}(\vartheta,\lambda)\bar{S}_{\ell m}(\vartheta',\lambda')\Big) \quad .$$

The decomposition formula or addition theorem for spherical harmonics (cf. e.g. (Moritz, 1980, p. 23 (3-30)))

$$P_\ell\big((\vartheta,\lambda);\,(\vartheta',\lambda')\big)= \\ \frac{1}{2\ell+1}\sum_{m=0}^{\ell}\big(\bar{C}_{\ell m}(\vartheta,\lambda)\bar{C}_{\ell m}(\vartheta',\lambda')+\bar{S}_{\ell m}(\vartheta,\lambda)\bar{S}_{\ell m}(\vartheta',\lambda')\big)$$

allows the for further simplification

$$\mathrm{Cov}\left\{\mathcal{U}(\vartheta,\lambda),\mathcal{U}(\vartheta',\lambda')\right\}=\sum_{\ell=0}^{\infty}\sigma_\ell^2P_\ell\big((\vartheta,\lambda);\,(\vartheta',\lambda')\big)$$

where the function value of the Legendre polynial $P_\ell\big((\vartheta,\lambda);\,(\vartheta',\lambda')\big)$ depends only on the spherical distance $\cos\psi$ between (ϑ,λ) and (ϑ',λ')

$$\cos\psi=\cos\vartheta\cos\vartheta'+\sin\vartheta\sin\vartheta'\cos(\lambda-\lambda') \; . \tag{27}$$

Finally this results in

$$\mathrm{Cov}\left\{\mathcal{U}(\vartheta,\lambda),\mathcal{U}(\vartheta',\lambda')\right\}=\sum_{\ell=0}^{\infty}\sigma_\ell^2\,P_\ell(\cos\psi)=\mathrm{cov}(\psi,\sigma_\ell^2) \; .$$

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