# Filtering of Correlated Data

—

# Stochastical Considerations Within GOCE Data Processing


Wolf-Dieter Schuh

Institute for Geodesy and Geoinformation
University Bonn


**GOCE-Summerschool**

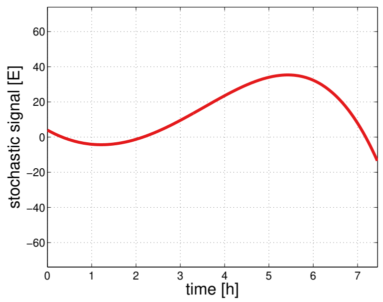
**Herrsching, $31^{th}$ May - $4^{th}$ June 2010**

# Contents

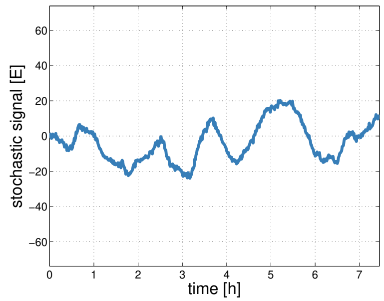
# Motivation

# The model

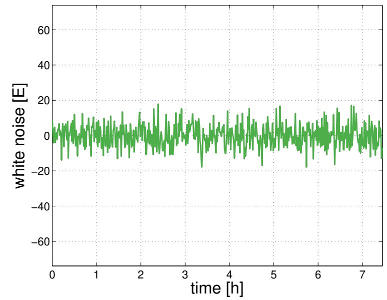





$f(x)$ ... deterministic model
($x$ ... parameter)



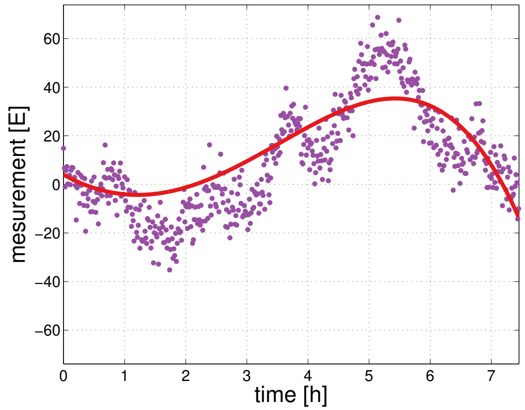

$s$ ... additional signal




$n$ ... white noise

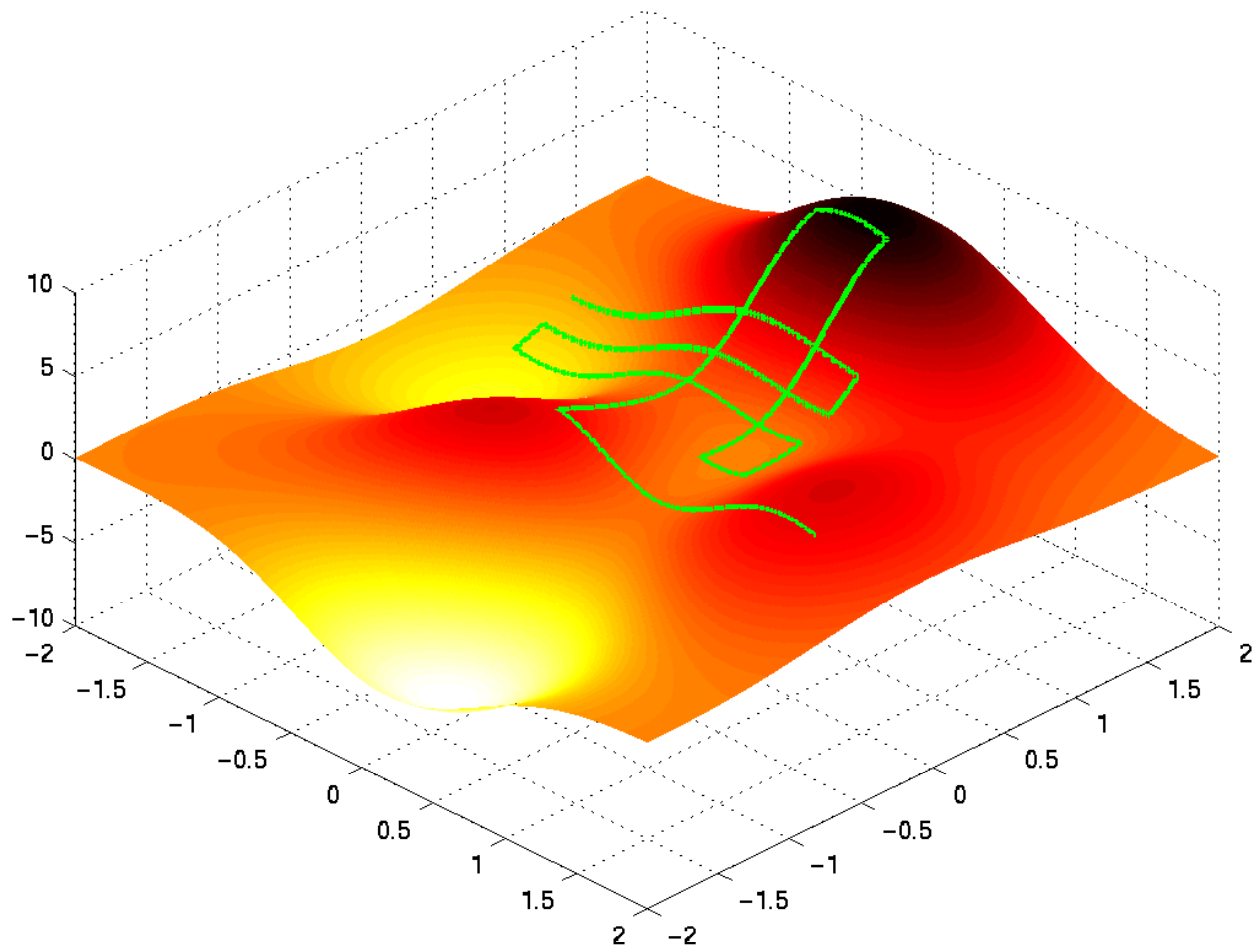
**measurements:**

$$\mathcal{L} = f(x) + s + \mathcal{N}$$



# Extended model






$$\mathcal{L} = f(x, x_{space}, x_{time}) + s(x_{space}, x_{time}) + \mathcal{N}$$

# Parameterization of the signal



$$\boxed{\mathcal{L} = f(x, x_s, x_t) + s(x_s, x_t) + \mathcal{N}}$$

$$\boxed{\mathcal{L} = f(x, x_s, x_t) + s(x_s, x_t) + \mathcal{N}}$$

- **external prior information**

  $\Delta\mathcal{L} = f(x, x_s, x_t) + \mathcal{N} \qquad \Delta\mathcal{L} = \mathcal{L} - s(x_s^{(p)}, x_t^{(p)})$

  - reduction strategies
  - de-aliasing

# Parameterization of the signal





$$\boxed{\mathcal{L} = f(x, x_s, x_t) + s(x_s, x_t) + \mathcal{N}}$$

- **external prior information**
  $\Delta\mathcal{L} = f(x, x_s, x_t) + \mathcal{N}$ $\quad\quad$ $\Delta\mathcal{L} = \mathcal{L} - s(x_s^{(p)}, x_t^{(p)})$
- **deterministic parameterization** - $x_2$ **nuisance variables**
  $\mathcal{L} = A_1 x_1 + A_2 x_2 + \mathcal{N}$
  - co-estimation (Schur-form, projection method)
  - selective infinite-variance weighting
  - short arc approach
  - empirical parameters, stochastic impulses

# Parameterization of the signal



$$\boxed{\mathcal{L} = f(x, x_s, x_t) + s(x_s, x_t) + \mathcal{N}}$$

- **external prior information**

  $\Delta\mathcal{L} = f(x, x_s, x_t) + \mathcal{N}$ $\qquad$ $\Delta\mathcal{L} = \mathcal{L} - s(x_s^{(p)}, x_t^{(p)})$
- **deterministic parameterization** - $x_2$ **nuisance variables**

  $\mathcal{L} = A_1 x_1 + A_2 x_2 + \mathcal{N}$
- **stochastic parameterization** - $\mathcal{S}$ **. . . random process**

  $\mathcal{L} = f(x, x_s, x_t) + \mathcal{S} + \mathcal{N}$ $\qquad$ $E\{\mathcal{S}\}, \Sigma\{\mathcal{S}\}$

  - Prediction, Collocation - Kernel functions
  - Wiener-Kolmogorov filtering - Covariance functions
  - Kriging - Variograms
  - Frequency selective metric - Linear discrete filters

| deterministic model | stochastic model |
|---|---|
| $\mathcal{L} = Ax + \mathcal{S} + \mathcal{N}$ | $E\{\mathcal{N}\} = 0,\ M\{E\{\mathcal{S}\}\} = 0$ |
| $\mathcal{V} = -\ \mathcal{S}\ -\ \mathcal{N}$ | $\Sigma\{\mathcal{N}\},\ \Sigma\{\mathcal{S}\},\ \Sigma\{\mathcal{S},\mathcal{N}\} = 0$ |
| optimization principle: $v^T\Sigma^{-1}v$ | |
| $\ell + v = Ax$ | $\Sigma = \Sigma\{\mathcal{S}\} + \Sigma\{\mathcal{N}\}$ |
| **estimations & variances** | **gridded data** |
| $\widetilde{x} = \left(A^T\Sigma^{-1}A\right)^{-1}A^T\Sigma^{-1}\ell$ <br> $\widetilde{s} = \Sigma\{\mathcal{S}\}\Sigma^{-1}(A\widetilde{x} - \ell)$ <br> $\Sigma\left\{\widetilde{\mathcal{X}}\right\}, \Sigma\left\{\widetilde{\mathcal{S}}\right\}$ | $\Sigma = \begin{bmatrix} c_0\,c_1 \ldots c_h & \ldots c_2\,c_1 \\ c_1\,c_0 \ldots & c_h \ldots c_3\,c_2 \\ \vdots & \ddots\ c_h \vdots \\ c_h & c_h \\ \vdots\ c_h \ddots & \vdots \\ c_2\,c_3 \ldots c_h & \ldots c_0\,c_1 \\ c_1\,c_2 \ldots & c_h \ldots c_1\,c_0 \end{bmatrix}$ |

# Complete decorrelation





| | | |
|---|---|---|
| input: | $\boldsymbol{\ell}, \quad \boldsymbol{\Sigma}$ | $\bar{\boldsymbol{\ell}} = \boldsymbol{F}\boldsymbol{\ell}, \quad \bar{\boldsymbol{\Sigma}} = \boldsymbol{I}$ |
| model: | $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\ell} + \boldsymbol{v}$ | $\bar{\boldsymbol{A}}\boldsymbol{x} = \bar{\boldsymbol{\ell}} + \bar{\boldsymbol{v}}$ |
| principle: | $\boldsymbol{v}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{v}$ | $\bar{\boldsymbol{v}}^T\bar{\boldsymbol{v}}$ |

$$\left(\boldsymbol{R}^{-1}\right)^T = \boldsymbol{F}$$

$$\boldsymbol{\Sigma} = \boldsymbol{R}^T\boldsymbol{R} \quad \Rightarrow$$

$$\bar{\boldsymbol{\ell}} = \left(\boldsymbol{R}^{-1}\right)^T \boldsymbol{\ell}$$

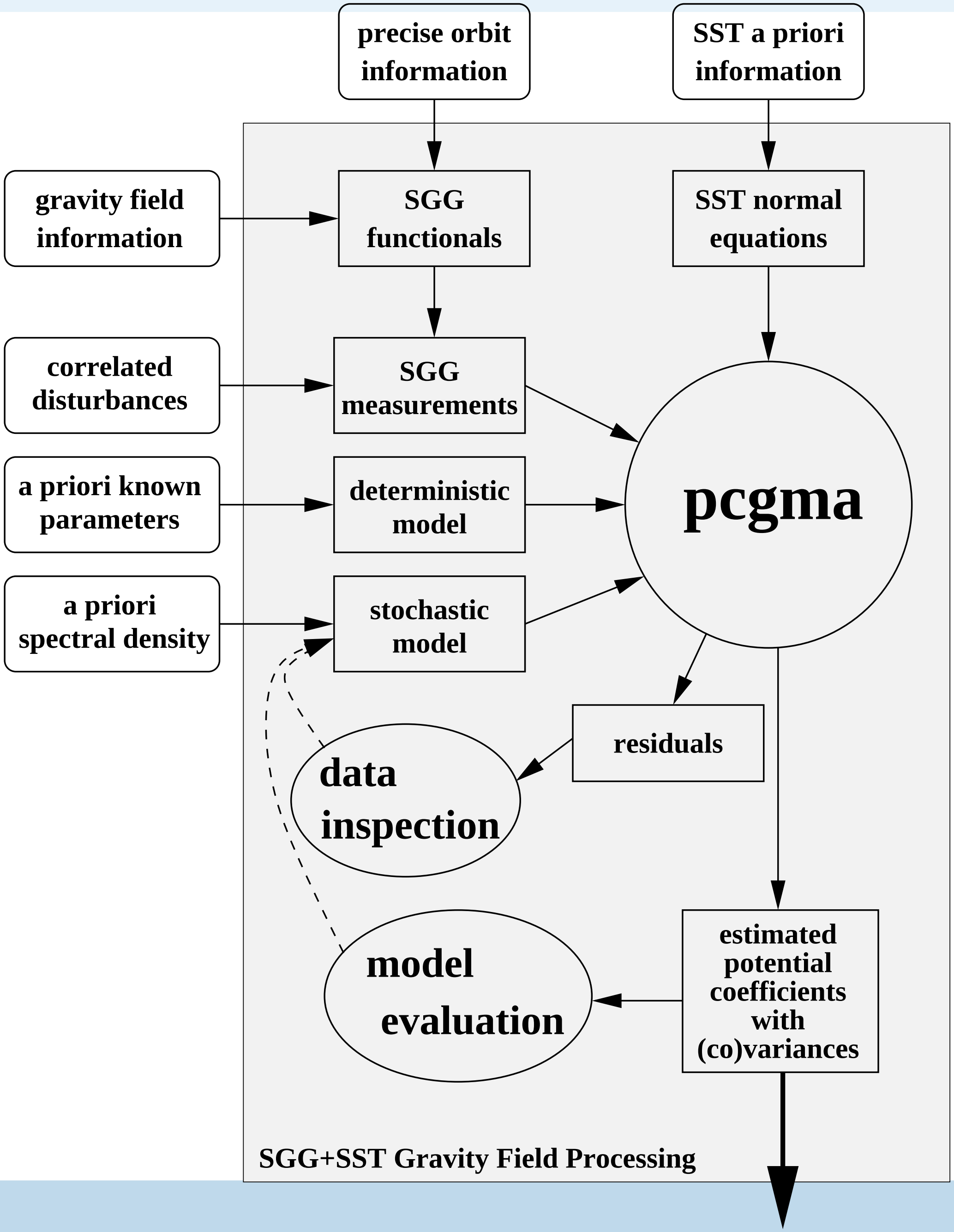
$$\bar{\boldsymbol{A}} = \left(\boldsymbol{R}^{-1}\right)^T \boldsymbol{A}$$

$$\bar{\boldsymbol{\Sigma}} = \left(\boldsymbol{R}^{-1}\right)^T \boldsymbol{\Sigma}\boldsymbol{R}^{-1} = \boldsymbol{I}$$

| **deterministic model** | **stochastic model** |
|---|---|
| $\mathcal{L} = Ax + \mathcal{S} + \mathcal{N}$ | $E\{\mathcal{N}\} = 0,\ M\{E\{\mathcal{S}\}\} = 0$ |
| $\mathcal{V} = \text{-}\ \mathcal{S}\ \text{-}\ \mathcal{N}$ | $\Sigma\{\mathcal{N}\},\ \Sigma\{\mathcal{S}\},\ \Sigma\{\mathcal{S},\mathcal{N}\} = 0$ |
| optimization principle: $v^T\Sigma^{-1}v$ | |
| $\ell + v = Ax$ | $\Sigma = \Sigma\{\mathcal{S}\} + \Sigma\{\mathcal{N}\}$ |
| **estimations & variances** | **gridded data** |
| $\widetilde{x} = \left(A^T\Sigma^{-1}A\right)^{-1}A^T\Sigma^{-1}\ell$ <br> $\widetilde{s} = \Sigma\{\mathcal{S}\}\,\Sigma^{-1}(A\widetilde{x} - \ell)$ <br> $\Sigma\left\{\widetilde{\mathcal{X}}\right\},\Sigma\left\{\widetilde{\mathcal{S}}\right\}$ | $\Sigma = \begin{bmatrix} c_0\, c_1 \ldots c_h & \ldots c_2\, c_1 \\ c_1\, c_0 \ldots & c_h \ldots c_3\, c_2 \\ \vdots & \ddots\ c_h \vdots \\ c_h & c_h \\ \vdots\ c_h \ddots & \vdots \\ c_2\, c_3 \ldots c_h & \ldots c_0\, c_1 \\ c_1\, c_2 \ldots & c_h \ldots c_1\, c_0 \end{bmatrix}$ |

# Residual analysis





# GOCE signal - colored noise

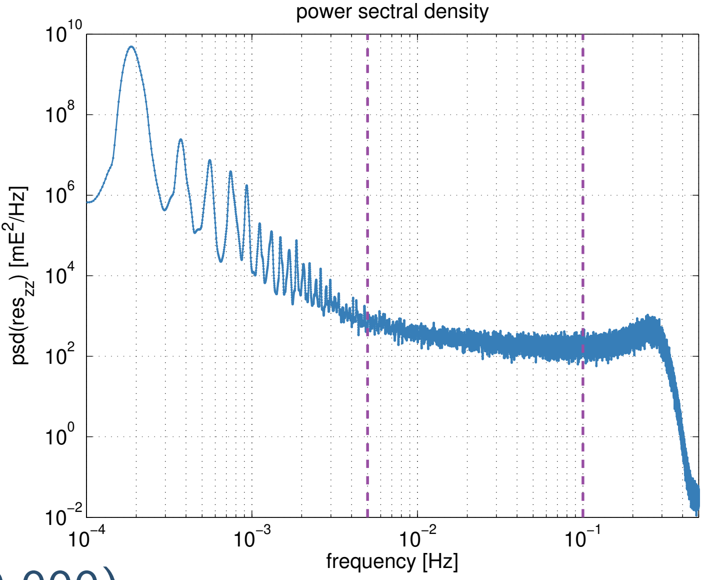




## Definition:

Correlated measurements

- white noise in the measurement bandwidth (0.005 Hz -0.1 Hz)
- $1/f$ characteristic with individual peaks outside the bandwidth




## Problems with GOCE data:

- number of measurements (100.000.000)
- long correlation length (1/rev)

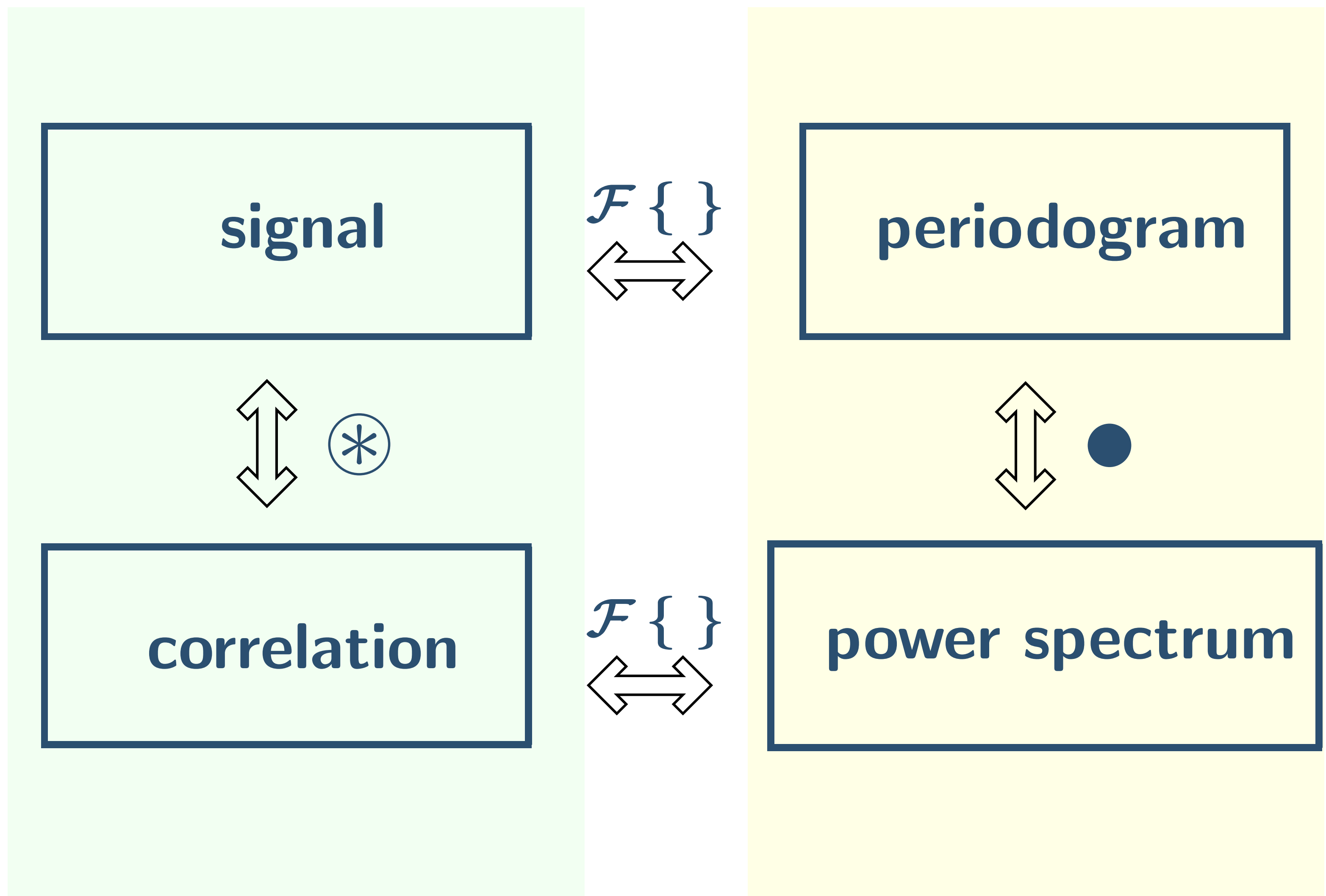
## Philosophy: tailored approach

- time-wise approach
- massive parallel computation
- discrete digital filters
- adaptive filter characteristic

# Magic Square

# Magic Square






time domain

frequency domain

# Magic square - deterministic signal



$$\boxed{f(t)}$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi t\nu} dt$$

$$\rightleftharpoons$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi t\nu} d\nu$$

$$\boxed{F(\nu)}$$

$$\Updownarrow \; f(t) \circledast f(t) = \int_{-\infty}^{\infty} f(u) f(u+t) du$$

$$\Updownarrow \; F(\nu) \bullet F(\nu)^*$$

$$\boxed{cor(t)}$$

$$psd(\nu) = \int_{-\infty}^{\infty} cor(t) \cos(2\pi t\nu) dt$$

$$\rightleftharpoons$$

$$cor(t) = \int_{-\infty}^{\infty} psd(\nu) \cos(2\pi t\nu) d\nu$$

$$\boxed{psd(\nu)}$$

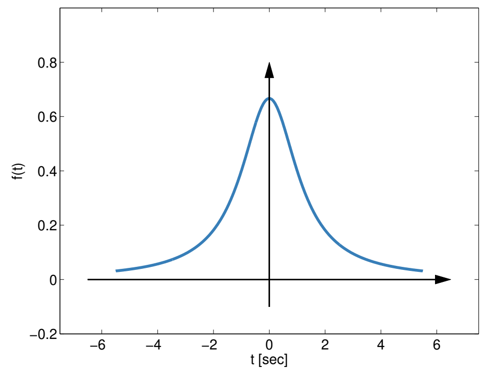
time domain

frequency domain

# Transition: continuous - discrete

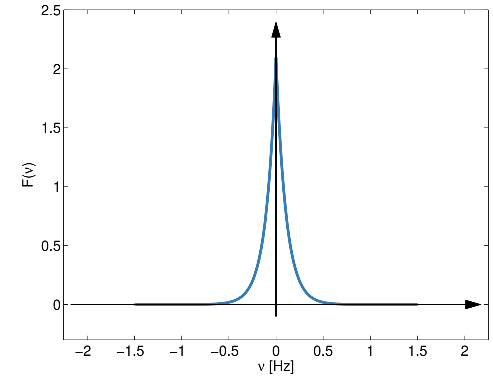
# Transition: continuous - discrete





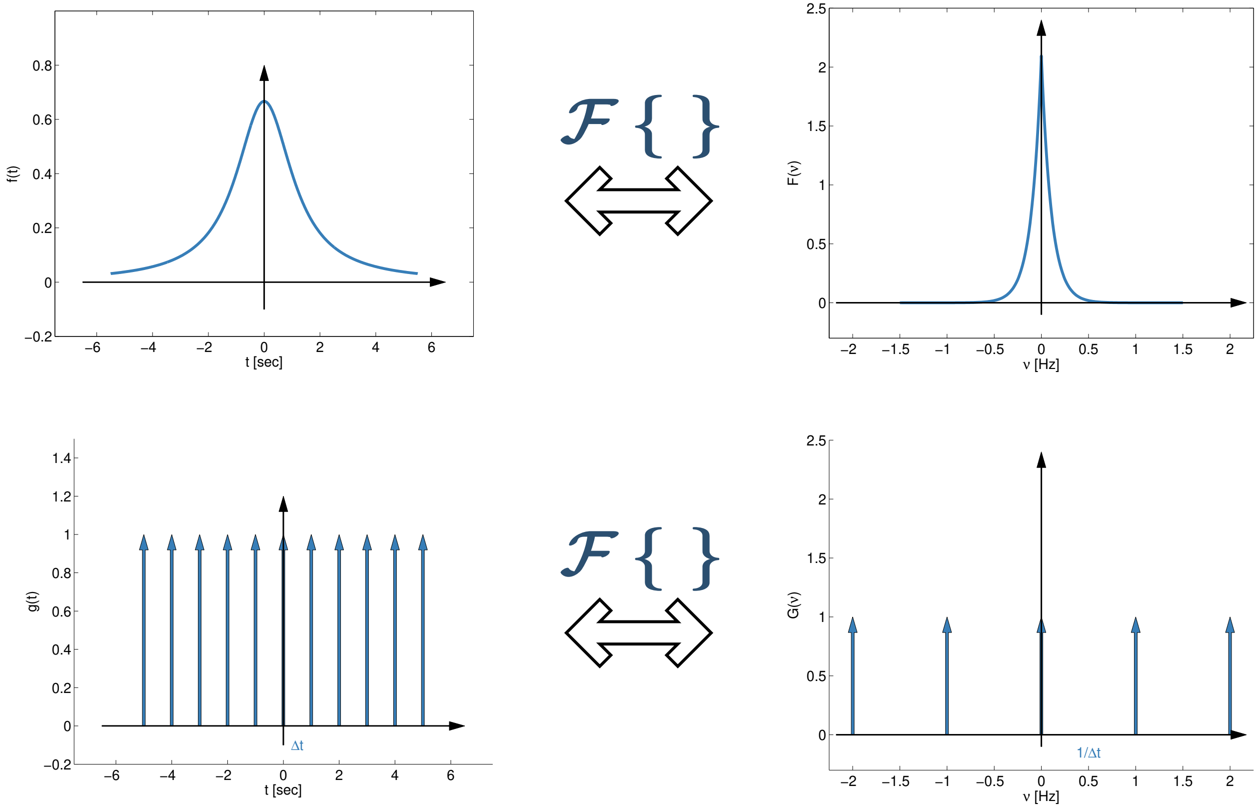

$\mathcal{F}\{\ \}$

$\Longleftrightarrow$



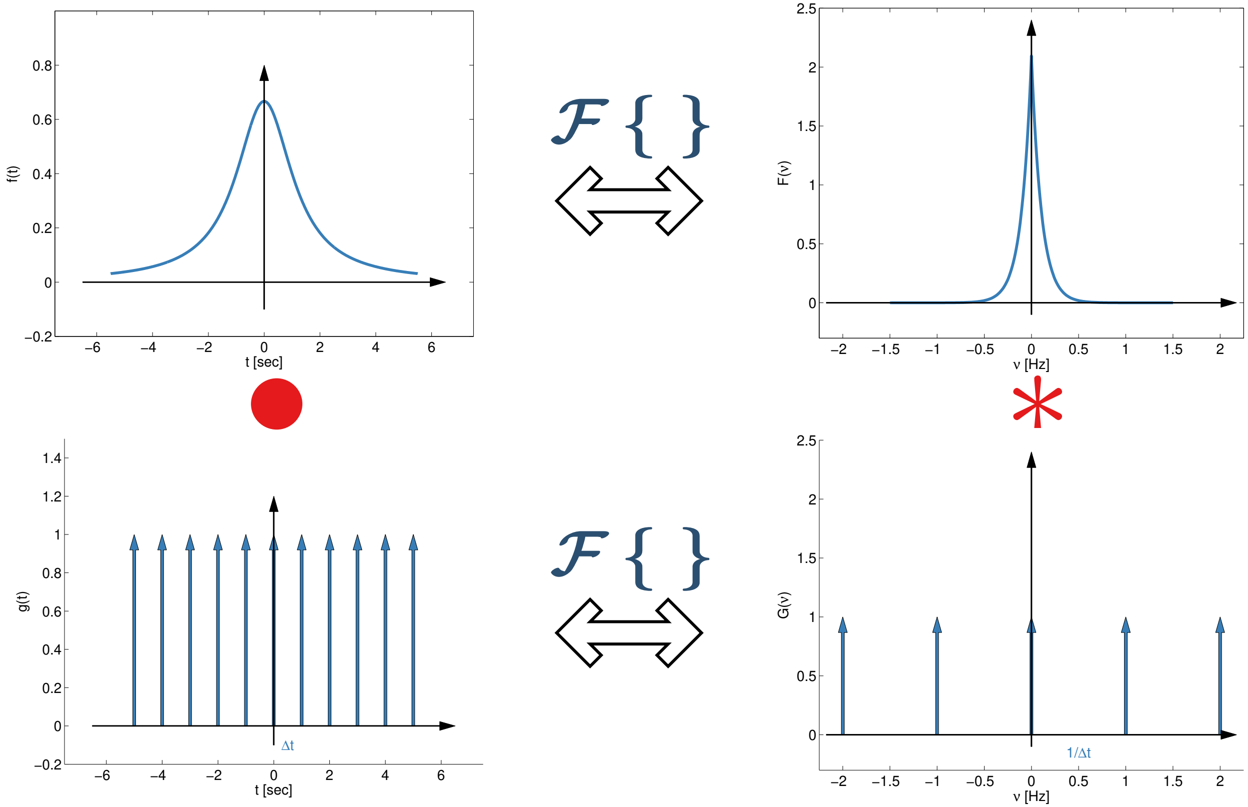
# Transition: continuous - discrete





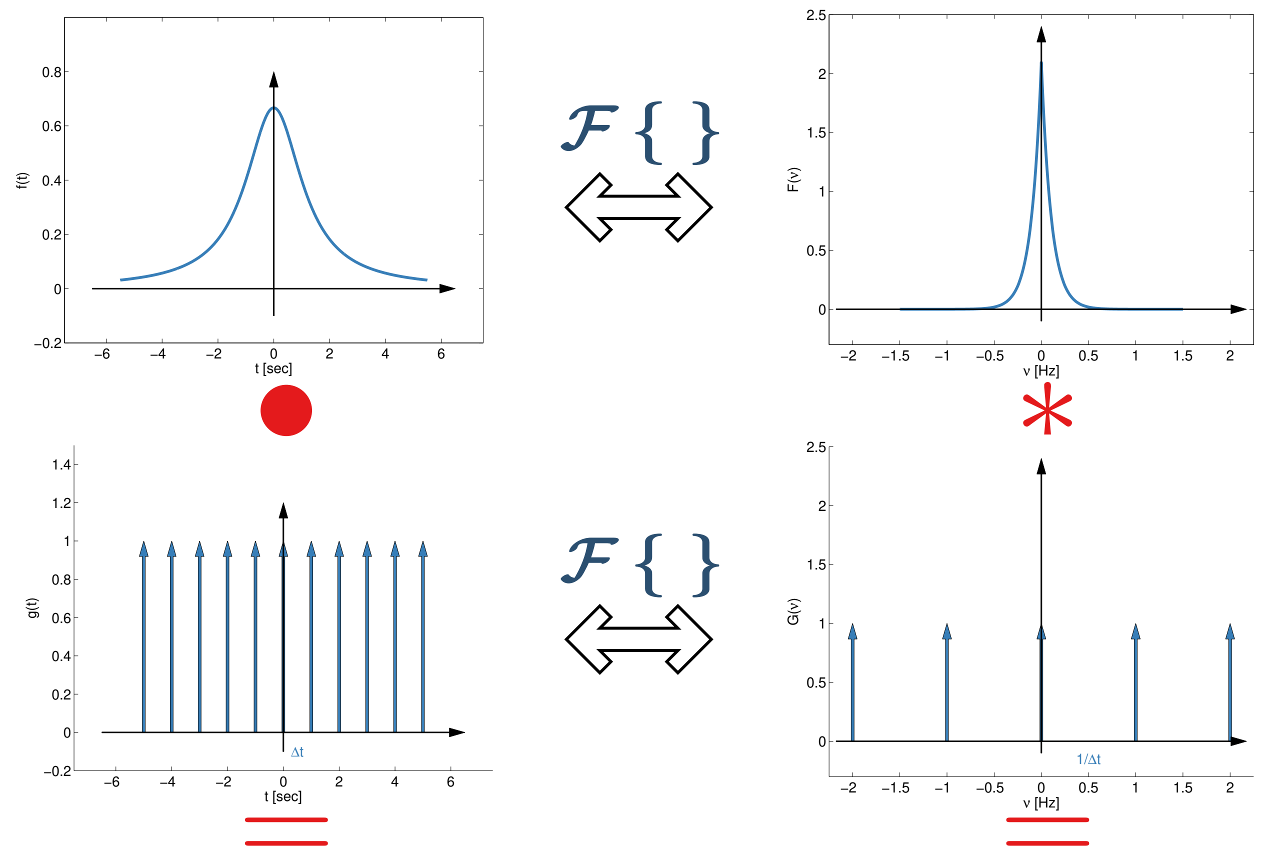
# Transition: continuous - discrete





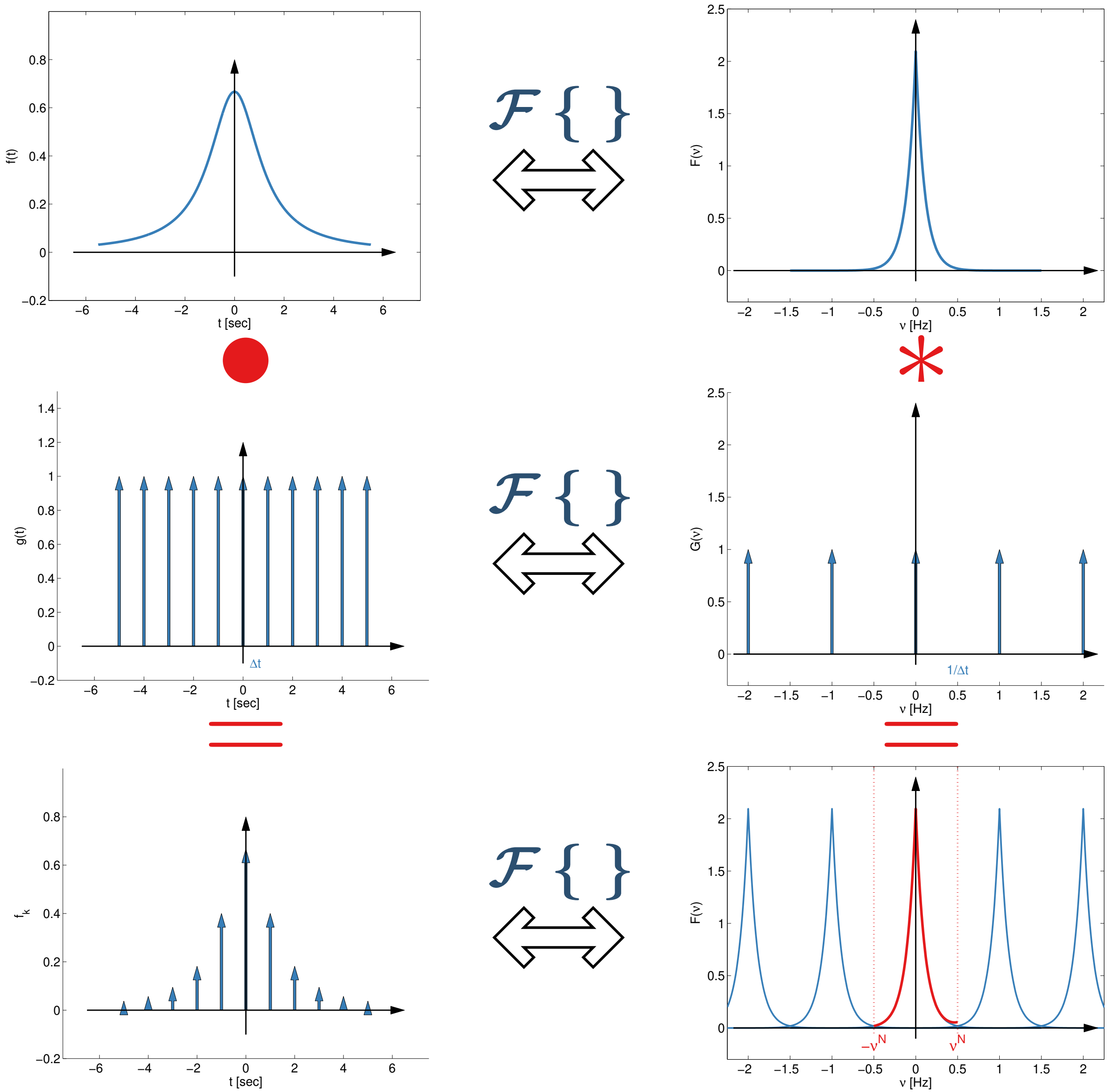
# Transition: continuous - discrete





# Transition: continuous - discrete





# Transition: continuous - discrete



$$\boxed{f(t)}$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi t\nu} dt$$

$$\rightleftharpoons$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi t\nu} d\nu$$

$$\boxed{F(\nu)}$$

time domain

frequency domain

# Transition: continuous - discrete

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi t\nu} dt$$

$$\boxed{f(t)} \quad \rightleftharpoons \quad \boxed{F(\nu)}$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi t\nu} d\nu$$

$$g(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)$$

$$G(\nu) = \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} \delta(\nu - \frac{n}{\Delta t})$$

time domain

frequency domain

# Transition: continuous - discrete



$$\boxed{f(t)}$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi t\nu} dt$$

$$\rightleftharpoons$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi t\nu} d\nu$$

$$\boxed{F(\nu)}$$

$$\bullet$$

$$*$$

$$g(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)$$

$$G(\nu) = \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} \delta(\nu - \frac{n}{\Delta t})$$

time domain

frequency domain

# Transition: continuous - discrete



$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi t\nu} dt$$

$$\boxed{f(t)} \quad \rightleftharpoons \quad \boxed{F(\nu)}$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi t\nu} d\nu$$

$$\bullet \qquad\qquad\qquad * $$

$$g(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \qquad\qquad G(\nu) = \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} \delta(\nu - \frac{n}{\Delta t})$$

$$= \qquad\qquad\qquad =$$

time domain

frequency domain

# Transition: continuous - discrete



$$\boxed{f(t)}$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi t\nu} dt$$

$$\rightleftharpoons$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi t\nu} d\nu$$

$$\boxed{F(\nu)}$$

$$\bullet$$

$$*$$

$$g(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)$$

$$G(\nu) = \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} \delta(\nu - \frac{n}{\Delta t})$$

$$=$$

$$=$$

$$\boxed{\{f_k\}_{\Delta t}}$$

$$F(\nu) = \Delta t \sum_{k=-\infty}^{\infty} f_k e^{-i2\pi\nu k\Delta t}$$

$$\rightleftharpoons$$

$$f_k = \int_{-\nu^N}^{\nu^N} F(\nu) e^{i2\pi\nu k\Delta t} d\nu$$

$$\boxed{F(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

time domain

frequency domain

# Transition: continuous - discrete

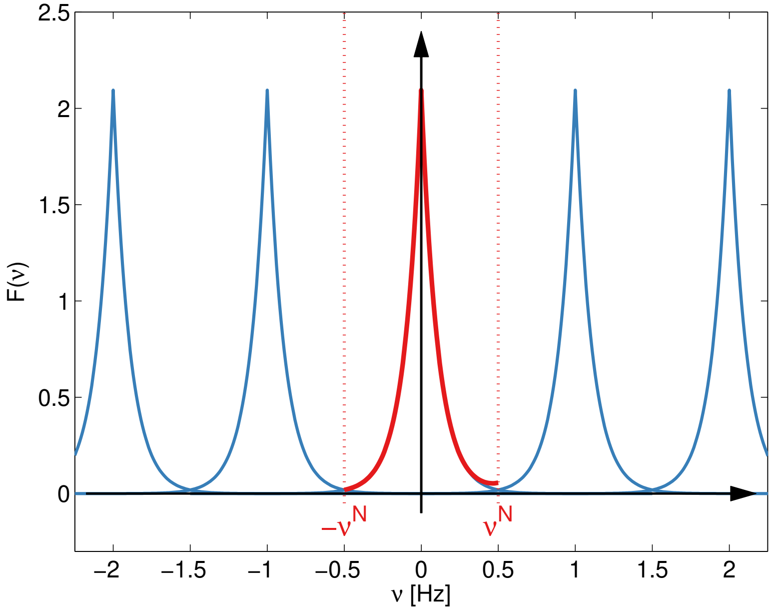


$$\boxed{F(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

**Fourier-transform of a discrete signal**

$$\boxed{F(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

**Fourier-transform of a discrete signal**



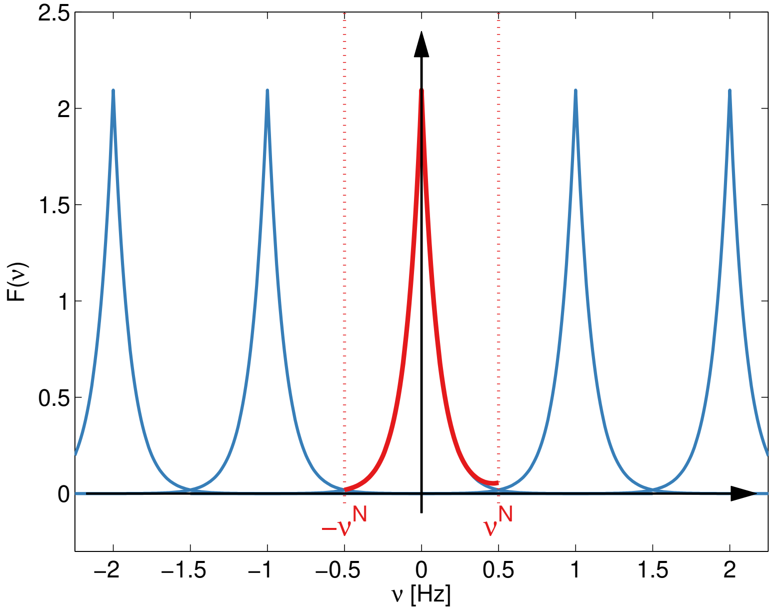
# Transition: continuous - discrete





$$\boxed{F(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

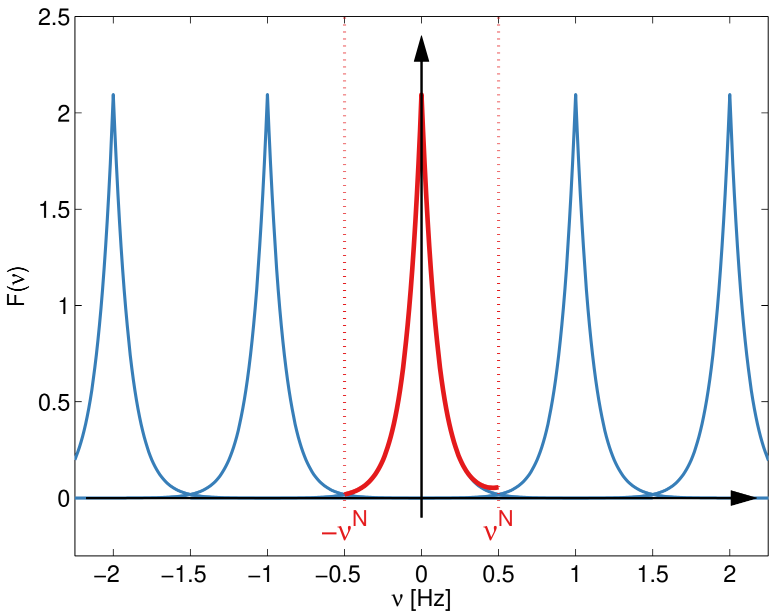
**Fourier-transform of a discrete signal**




**properties:**

$$\boxed{F(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

**Fourier-transform of a discrete signal**

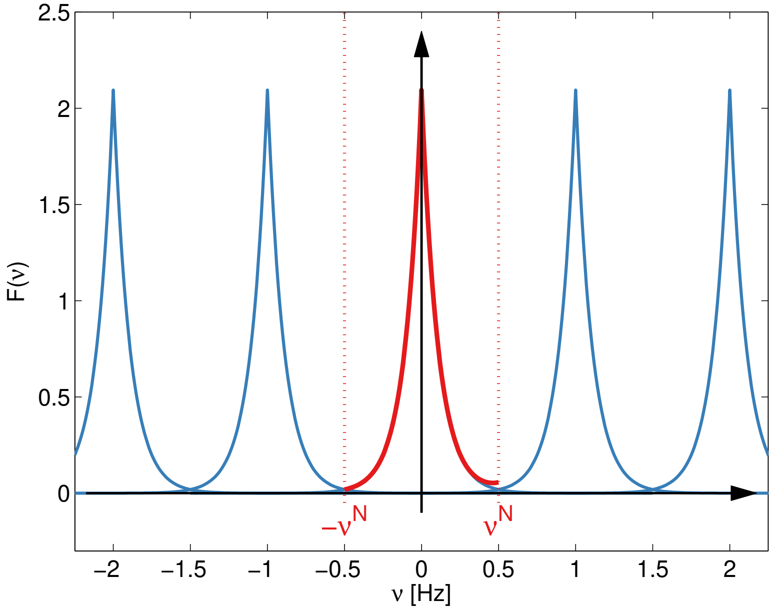



**properties:**

- **continuous**

$$\boxed{F(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

**Fourier-transform of a discrete signal**

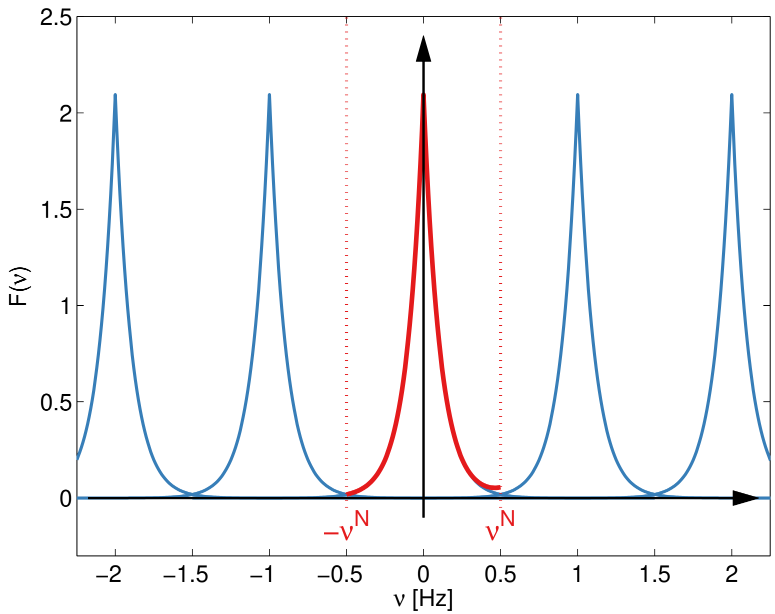



**properties:**

- **continuous**
- **periodic with** $\frac{1}{\Delta t}$

$$\boxed{F(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

**Fourier-transform of a discrete signal**




**properties:**

- **continuous**
- **periodic with $\frac{1}{\Delta t}$**
- **Nyquist frequency**

$$\nu^N = \frac{1}{2\Delta t}$$

# Magic square - deterministic discrete signal



$$\boxed{\{f_k\}_{\Delta t}}$$

$$F(\nu) = \Delta t \sum_{k=-\infty}^{\infty} f_k e^{-i2\pi\nu k\Delta t}$$

$$\rightleftharpoons$$

$$f_k = \int_{-\nu^N}^{\nu^N} F(\nu) e^{i2\pi\nu k\Delta t} d\nu$$

$$\boxed{F(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

$$\Updownarrow \{f_k\} \circledast \{f_k\} = \sum_{u=-\infty}^{\infty} f(u) f(u+k)$$

$$\Updownarrow \frac{1}{\Delta t} F(\nu) \bullet F(\nu)^*$$

$$\boxed{\{cor_k\}}$$

$$psd(\nu) = \Delta t \sum_{k=-\infty}^{\infty} cor_k \cos(2\pi\nu k\Delta t)$$

$$\rightleftharpoons$$

$$cor_k = \int_{-\nu^N}^{\nu^N} psd(\nu) \cos(2\pi\nu k\Delta t) d\nu$$

$$\boxed{psd(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

time domain

frequency domain

# Magic square - stochastic signal



$\mathcal{L}(t)$

$\Updownarrow$ $E\left\{(\mathcal{L}(u)-\mu)(\mathcal{L}(u+t)-\mu)\right\}$

$$S(\nu) = \int_{-\infty}^{\infty} \gamma(t)\cos(2\pi t\nu)dt$$

$$\gamma(t) \rightleftharpoons S(\nu)$$

$$\gamma(t) = \int_{-\infty}^{\infty} S(\nu)\cos(2\pi t\nu)d\nu$$

time domain

frequency domain

$$\{\ell_k\}_{\Delta t}$$

$$L(\nu) = \Delta t \sum_{k=-\infty}^{\infty} \ell_k e^{-i2\pi\nu k\Delta t}$$

$$\rightleftharpoons$$

$$\ell_k = \int_{-\nu^N}^{\nu^N} L(\nu) e^{i2\pi\nu k\Delta t} d\nu$$

$$L(\nu)\Big|_{-\nu^N}^{\nu^N}$$

$$\Updownarrow \; M\left\{\{\ell_k\} \circledast \{\ell_k\}\right\}$$

$$M\left\{\frac{1}{\Delta t} L(\nu) \bullet L(\nu)^*\right\} \; \Updownarrow$$

$$\{\widetilde{g}_k\}$$

$$\widetilde{psd}(\nu) = \Delta t \sum_{k=-\infty}^{\infty} \widetilde{g}_k \cos(2\pi\nu k\Delta t)$$

$$\rightleftharpoons$$

$$\widetilde{g}_k = \int_{-\nu^N}^{\nu^N} \widetilde{psd}(\nu) \cos(2\pi\nu k\Delta t) d\nu$$

$$\widetilde{psd}(\nu)\Big|_{-\nu^N}^{\nu^N}$$

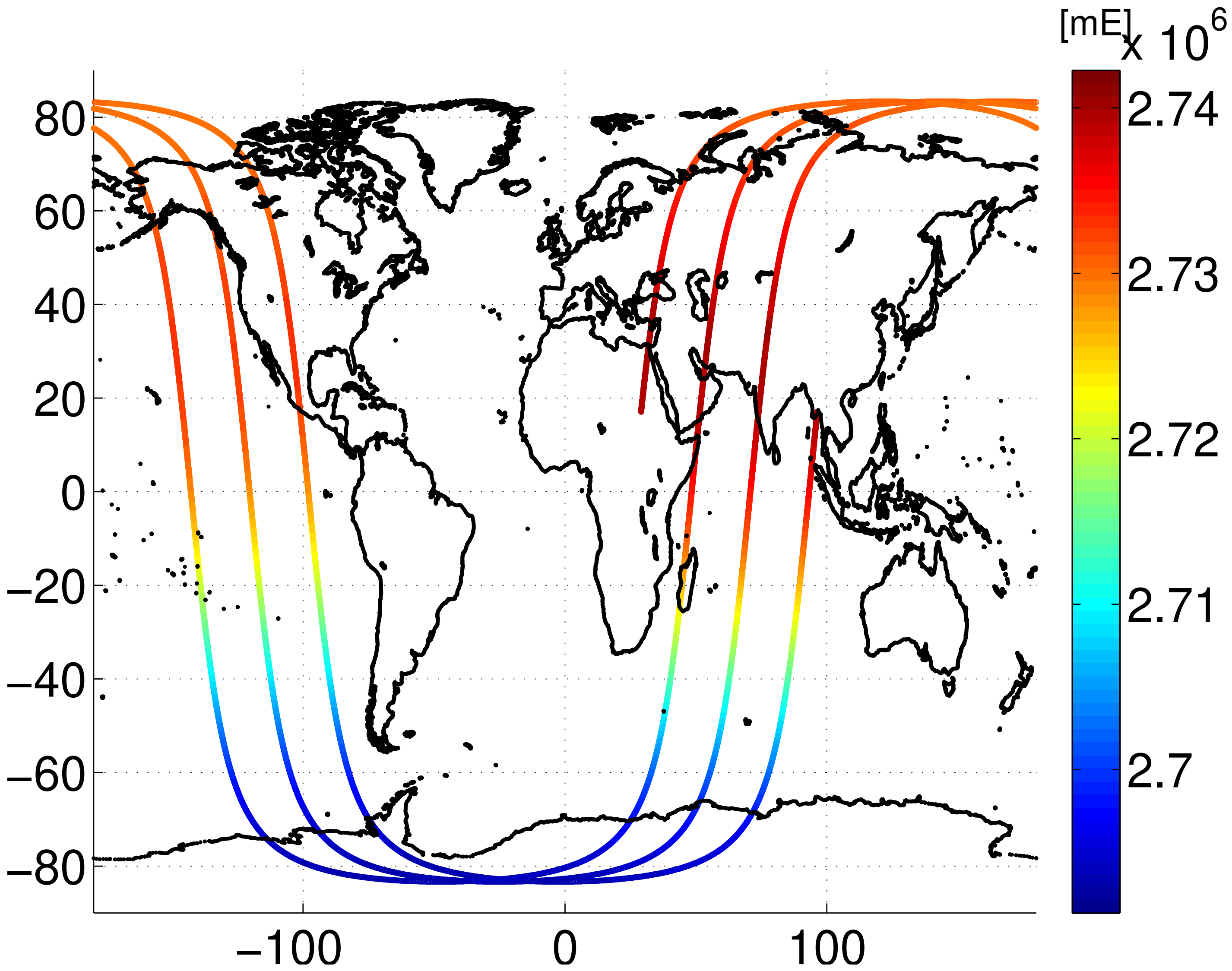
time domain

frequency domain

# GOCE data

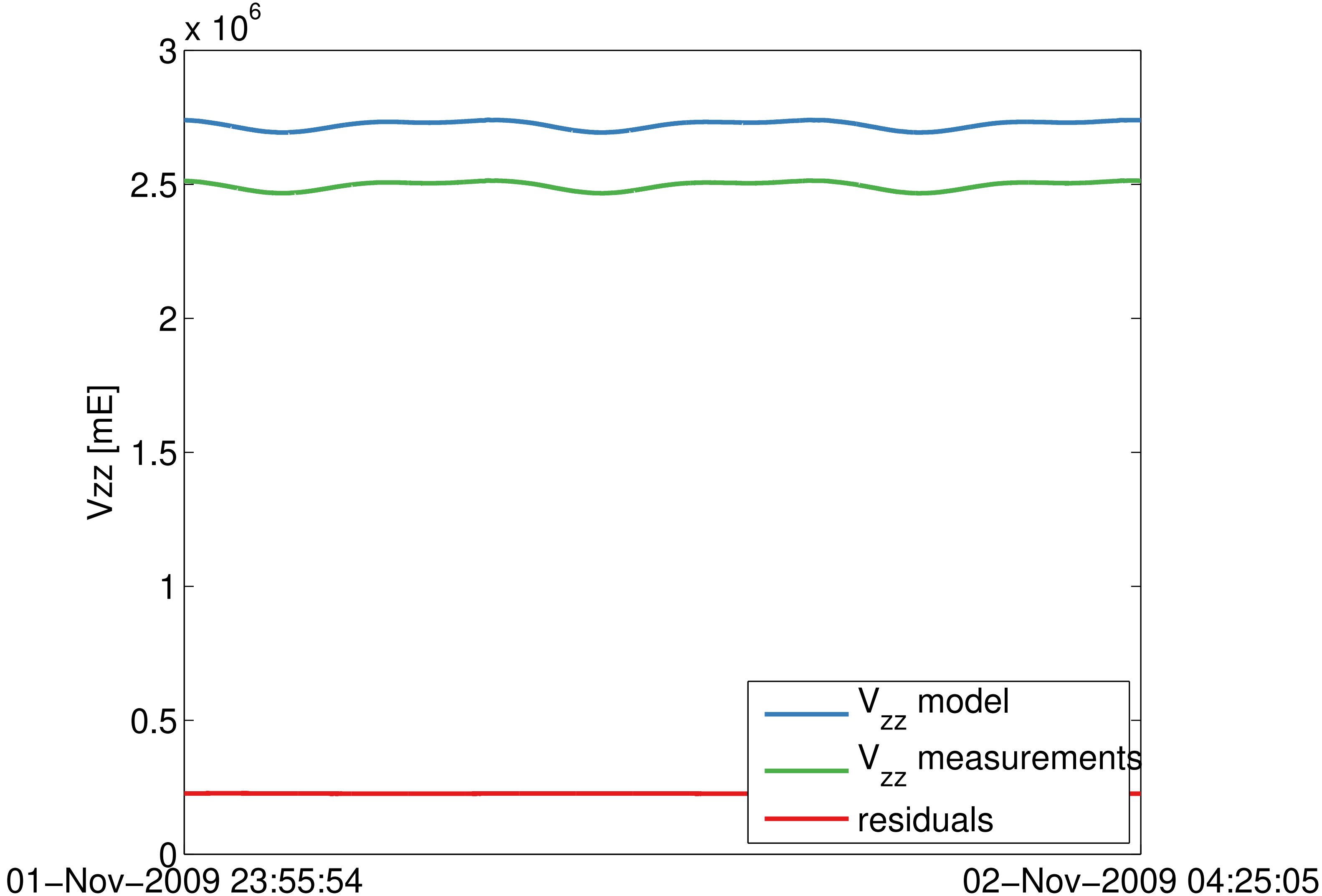
# GOCE - EGG-NOM_2 $V_{zz}$ signal





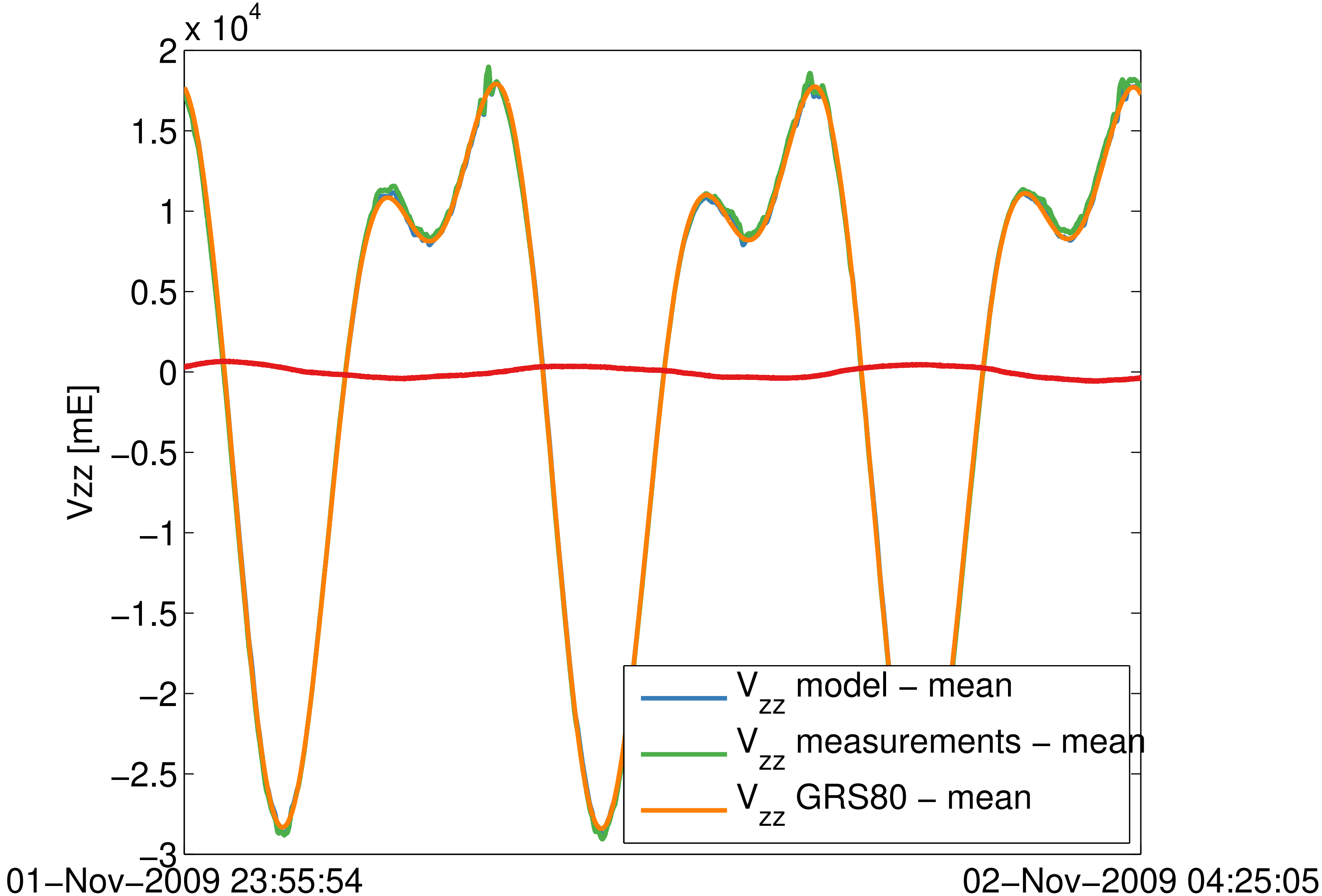
# GOCE - EGG-NOM_2 $V_{zz}$ signal





# GOCE - EGG-NOM_2 $V_{zz}$ signal

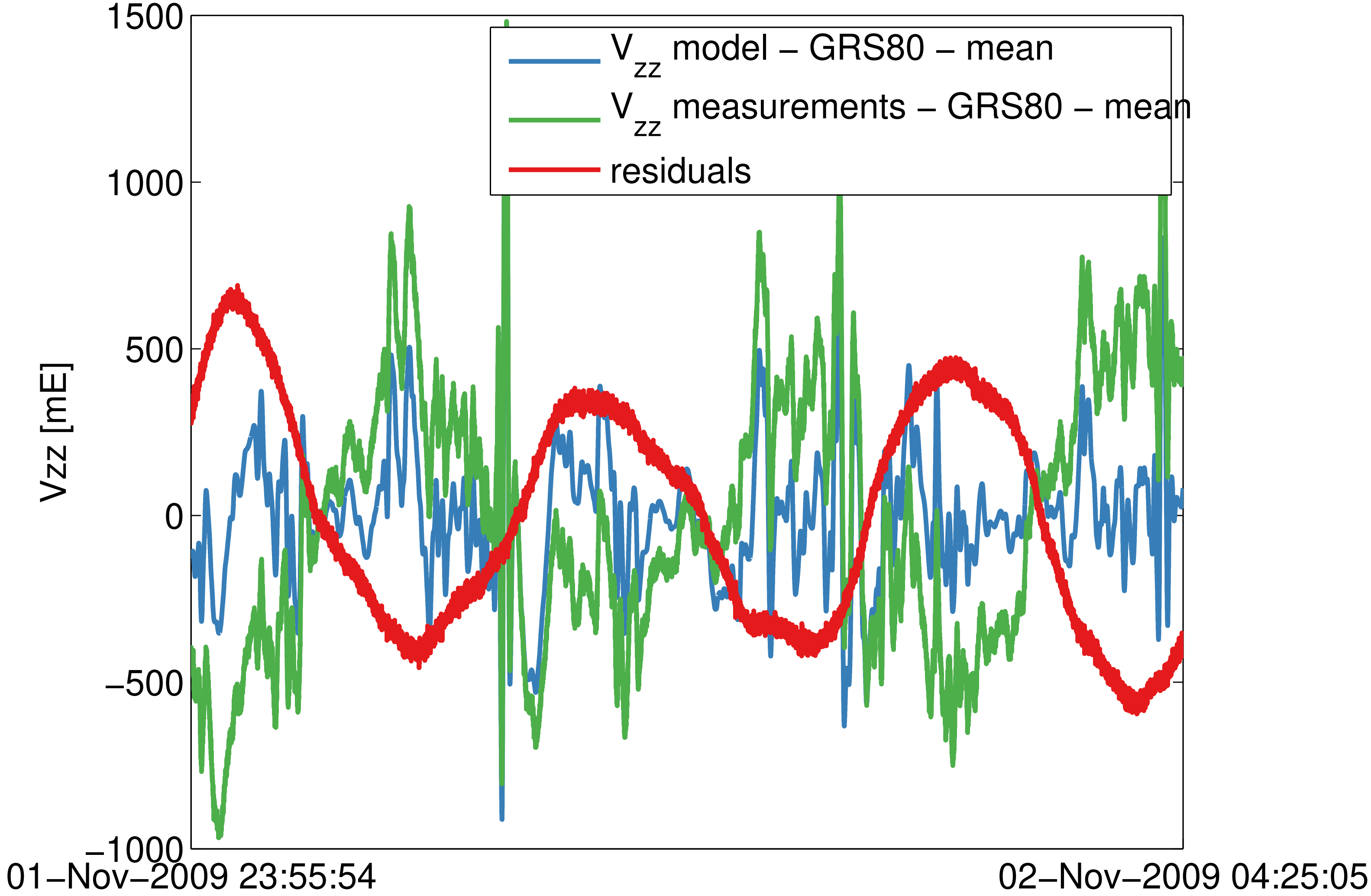




# GOCE - EGG-NOM_2 $V_{zz}$ signal

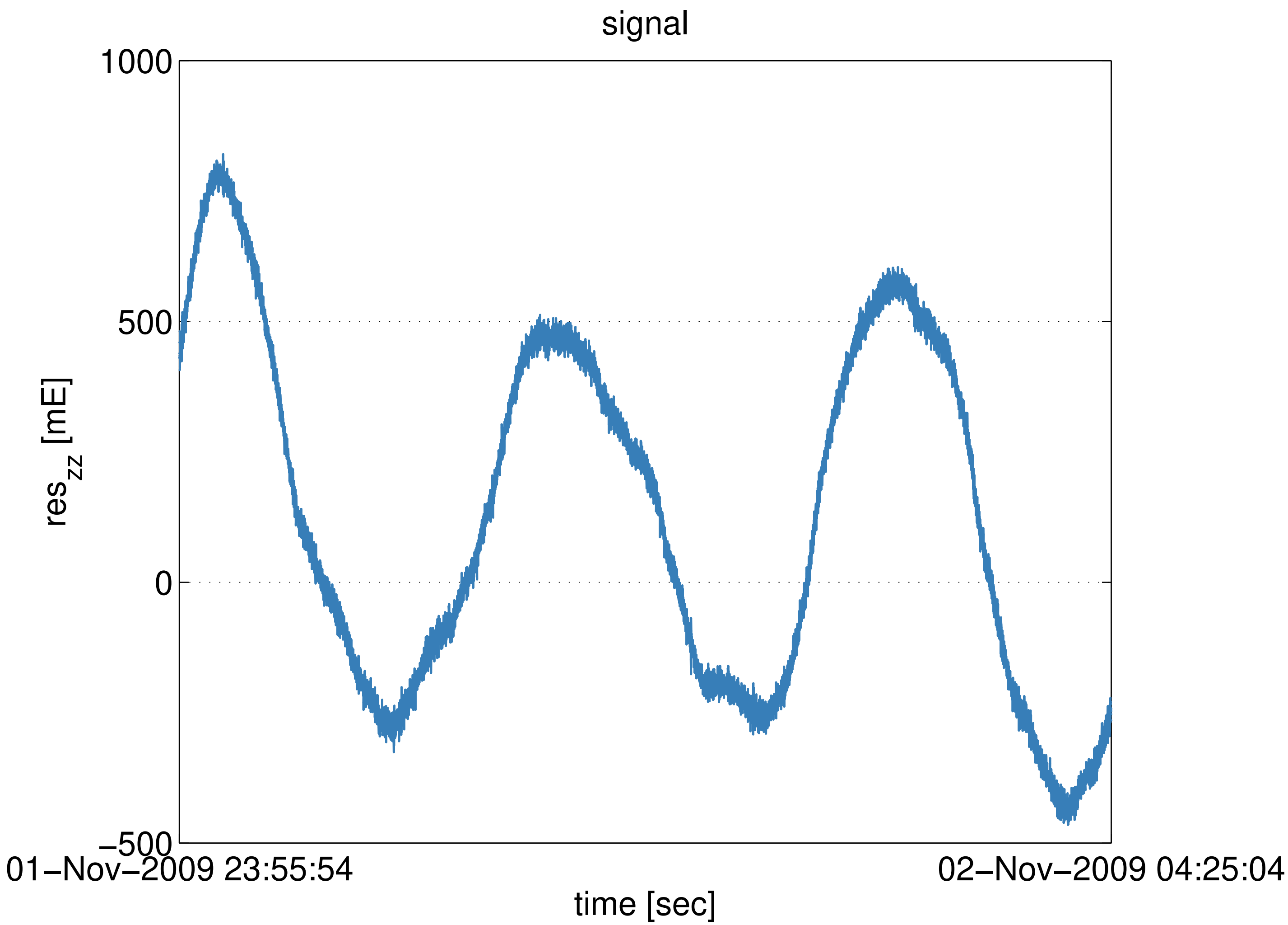






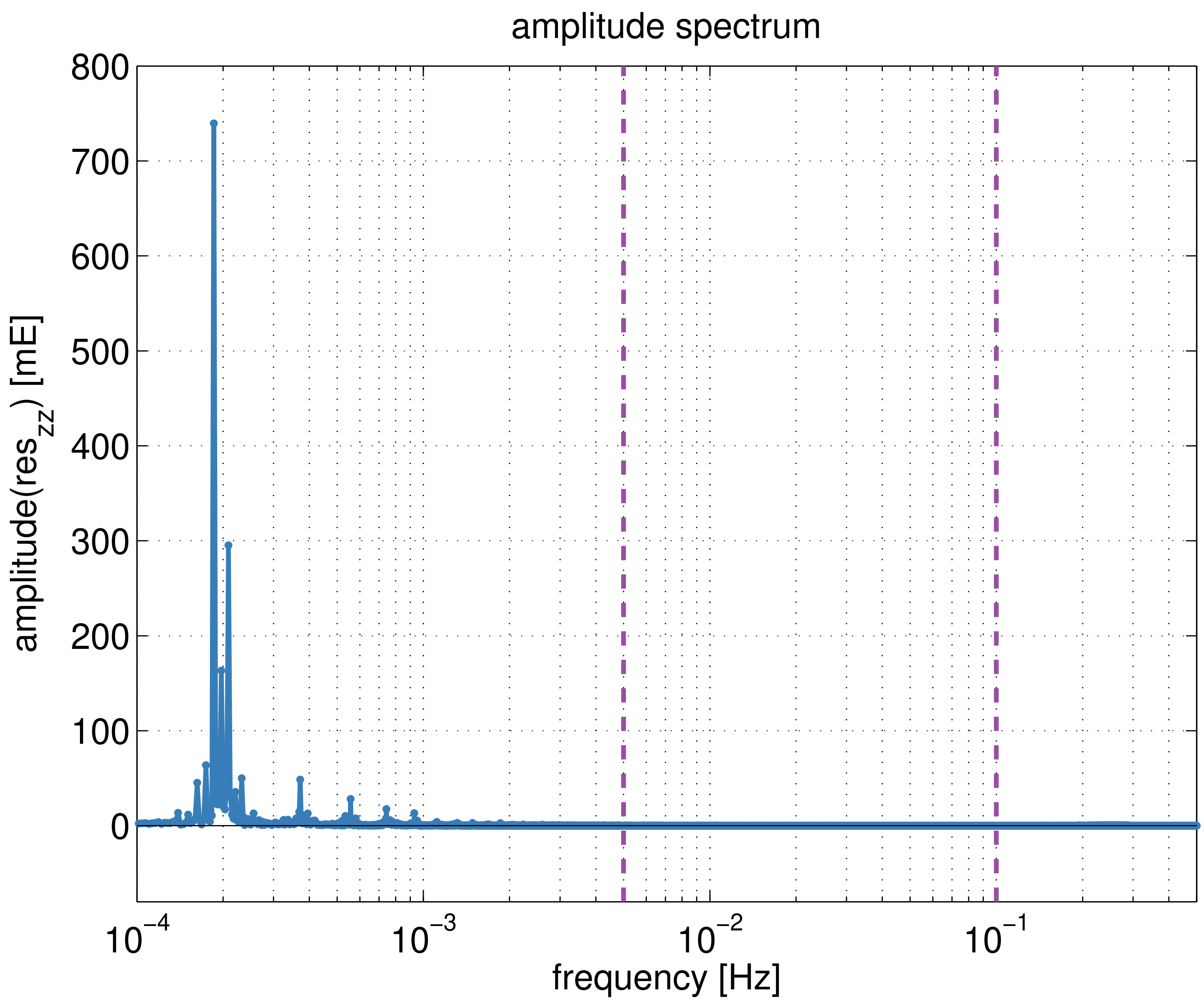
# GOCE - EGG-NOM_2 $V_{zz}$ residuals





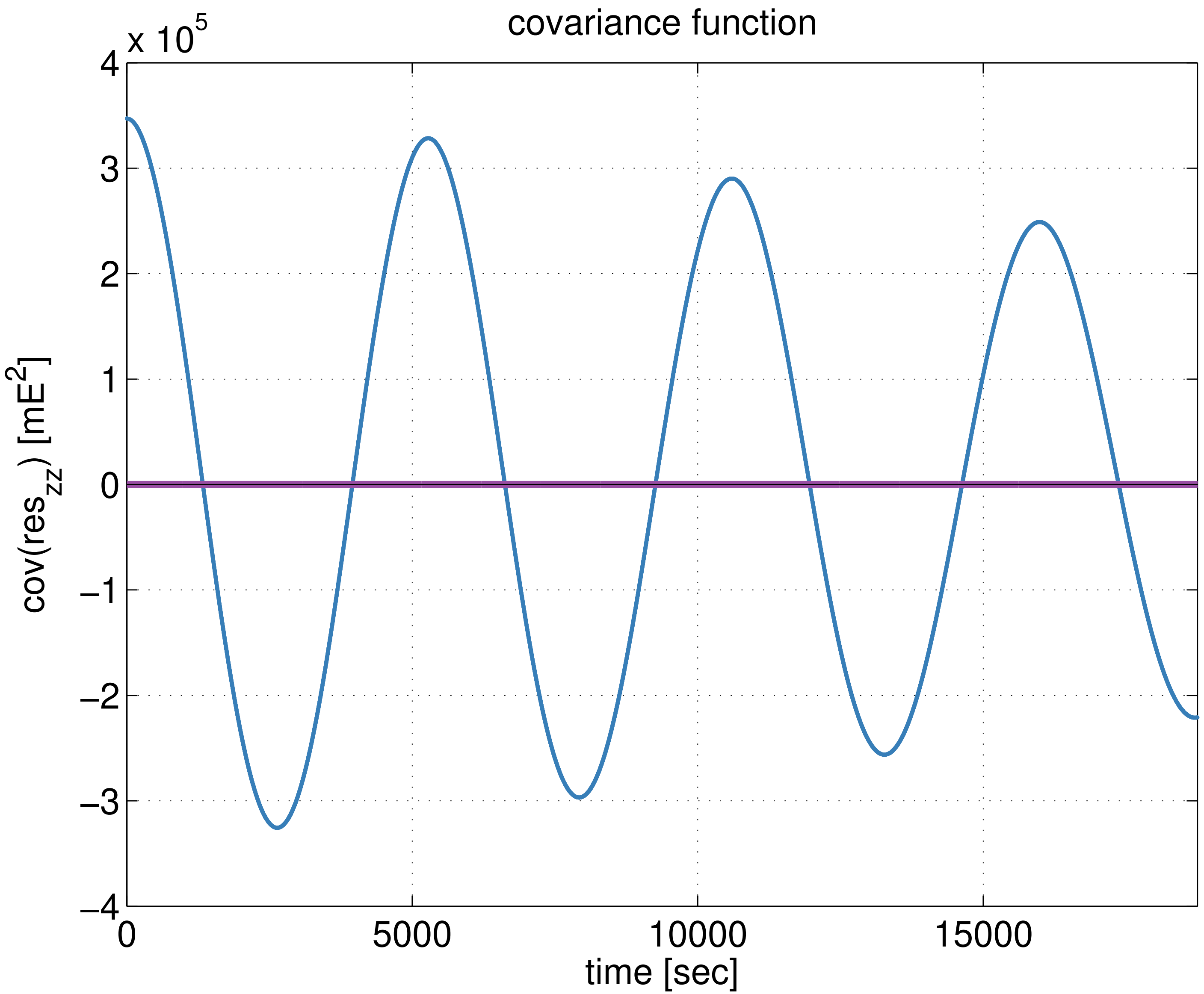
# GOCE - EGG-NOM_2 $V_{zz}$ residuals





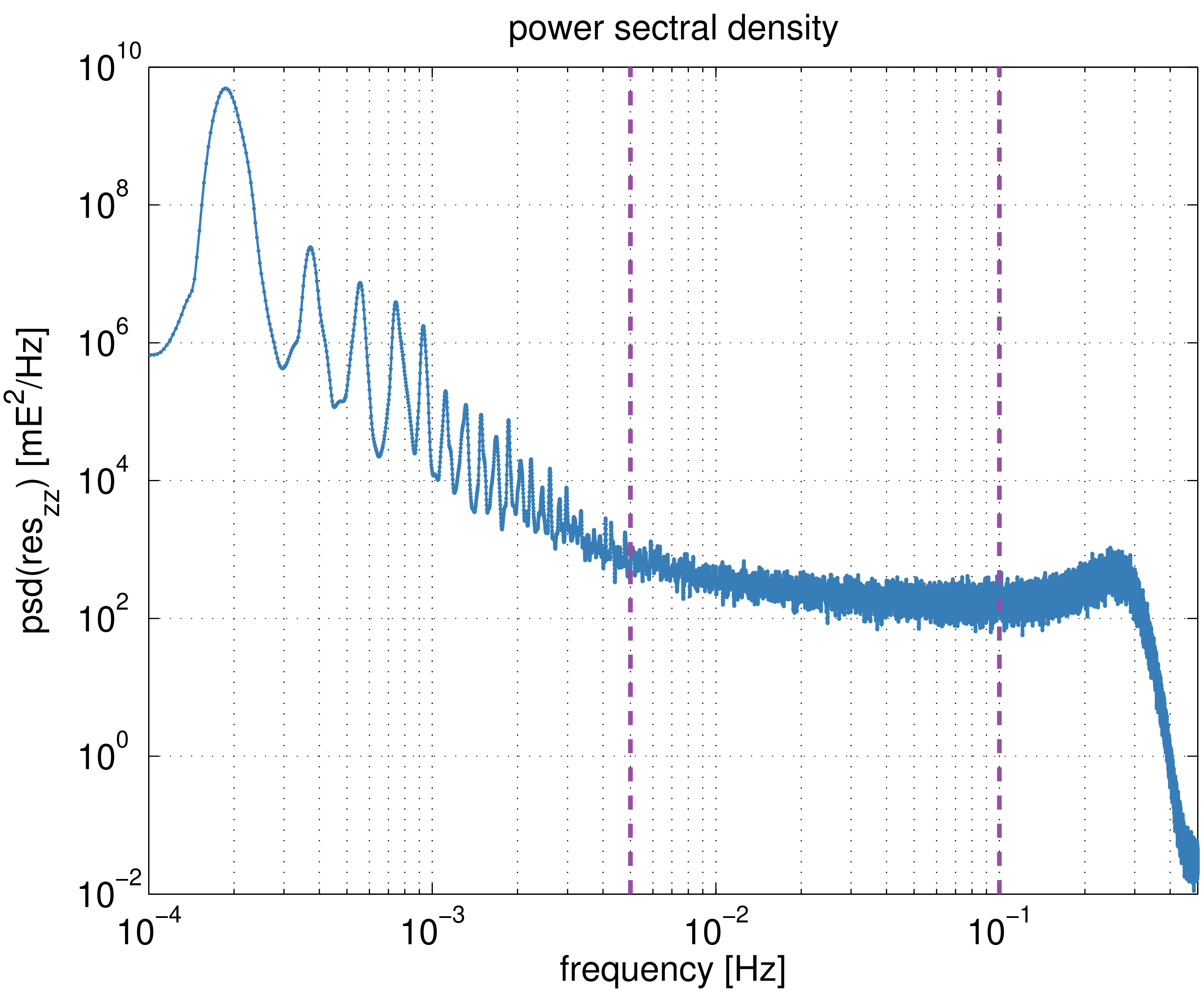
# GOCE - EGG-NOM_2 $V_{zz}$ residuals





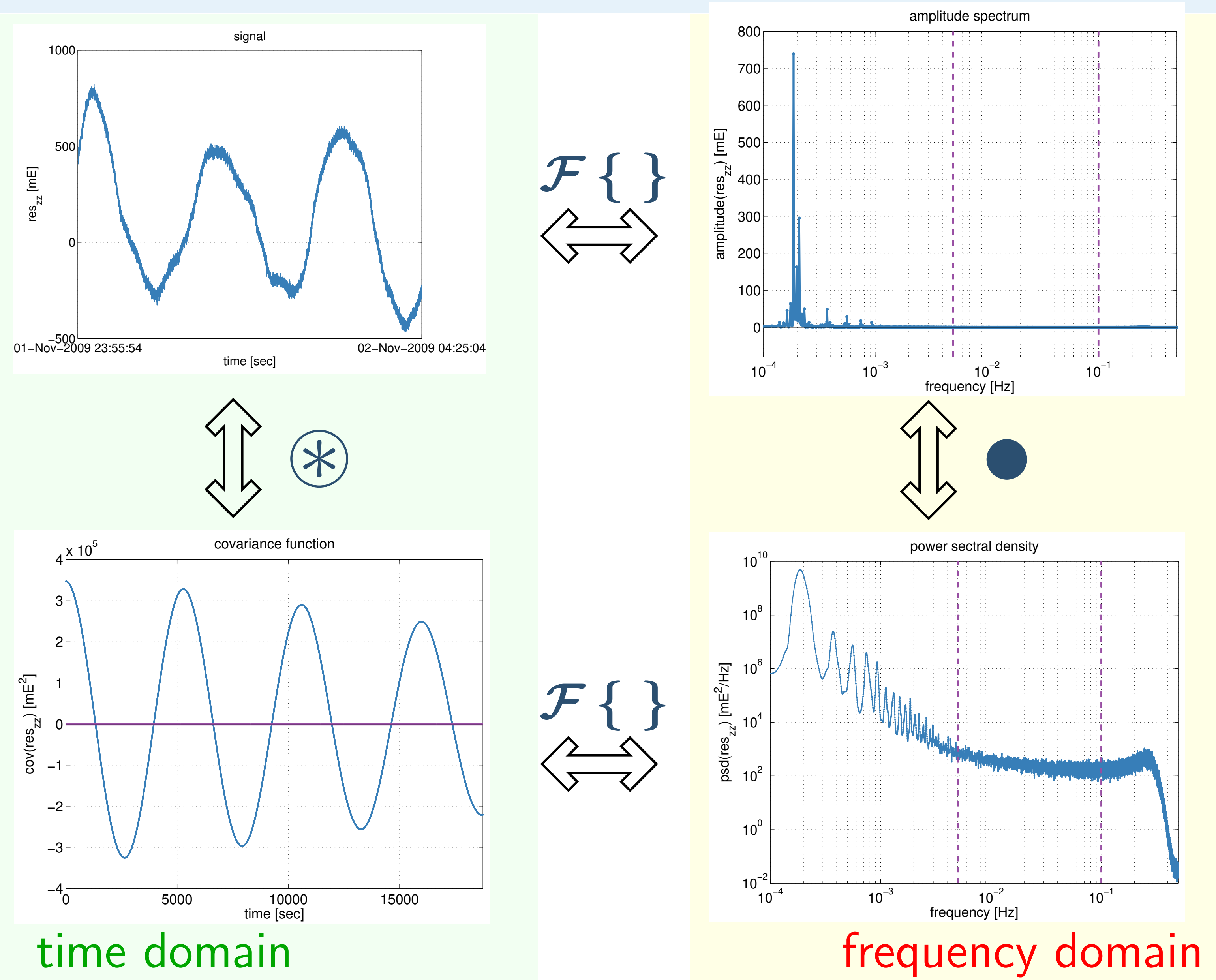
# GOCE - EGG-NOM_2 $V_{zz}$ residuals





# GOCE - EGG-NOM_2 $V_{zz}$ residuals

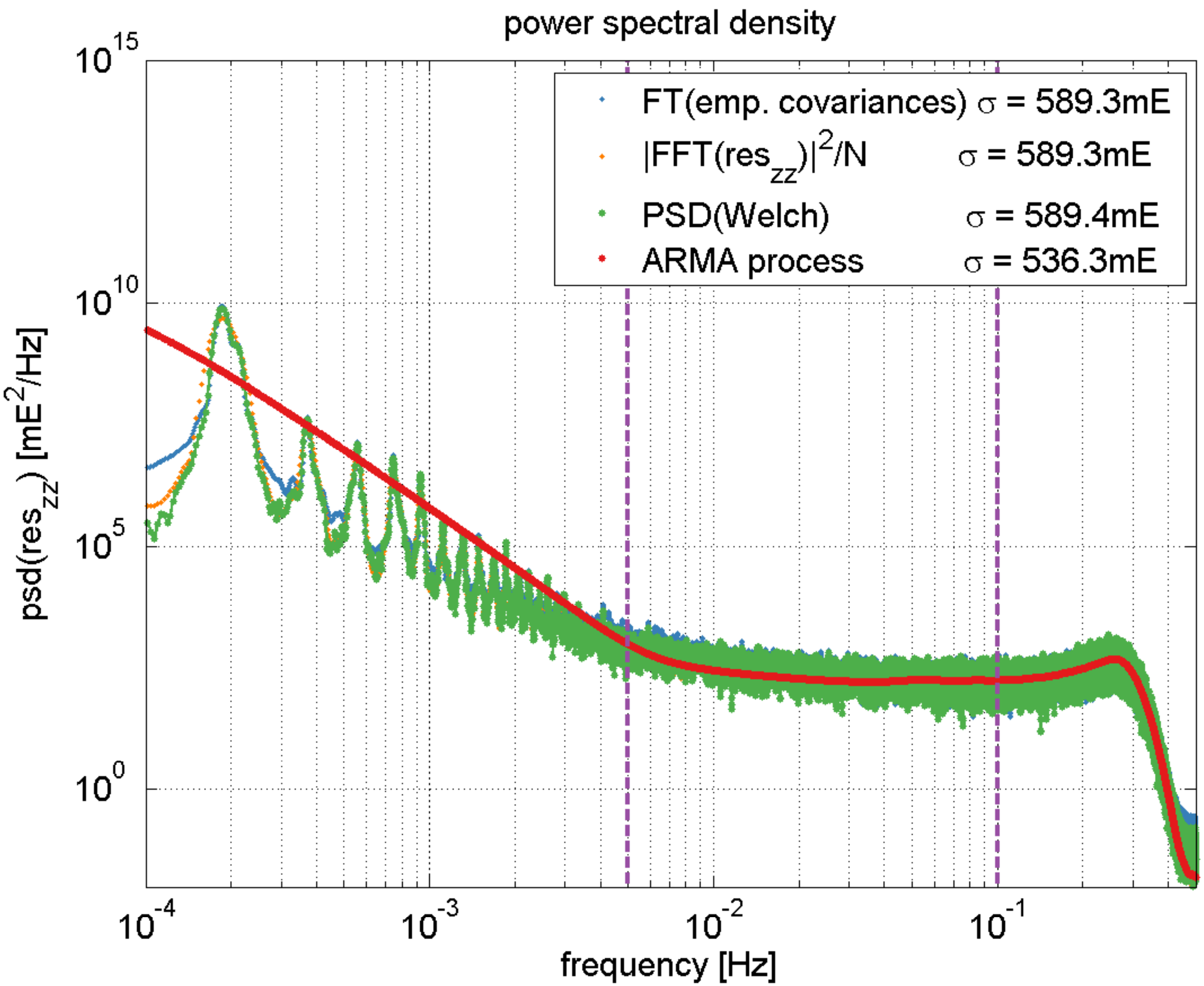




# GOCE - power spectrum density

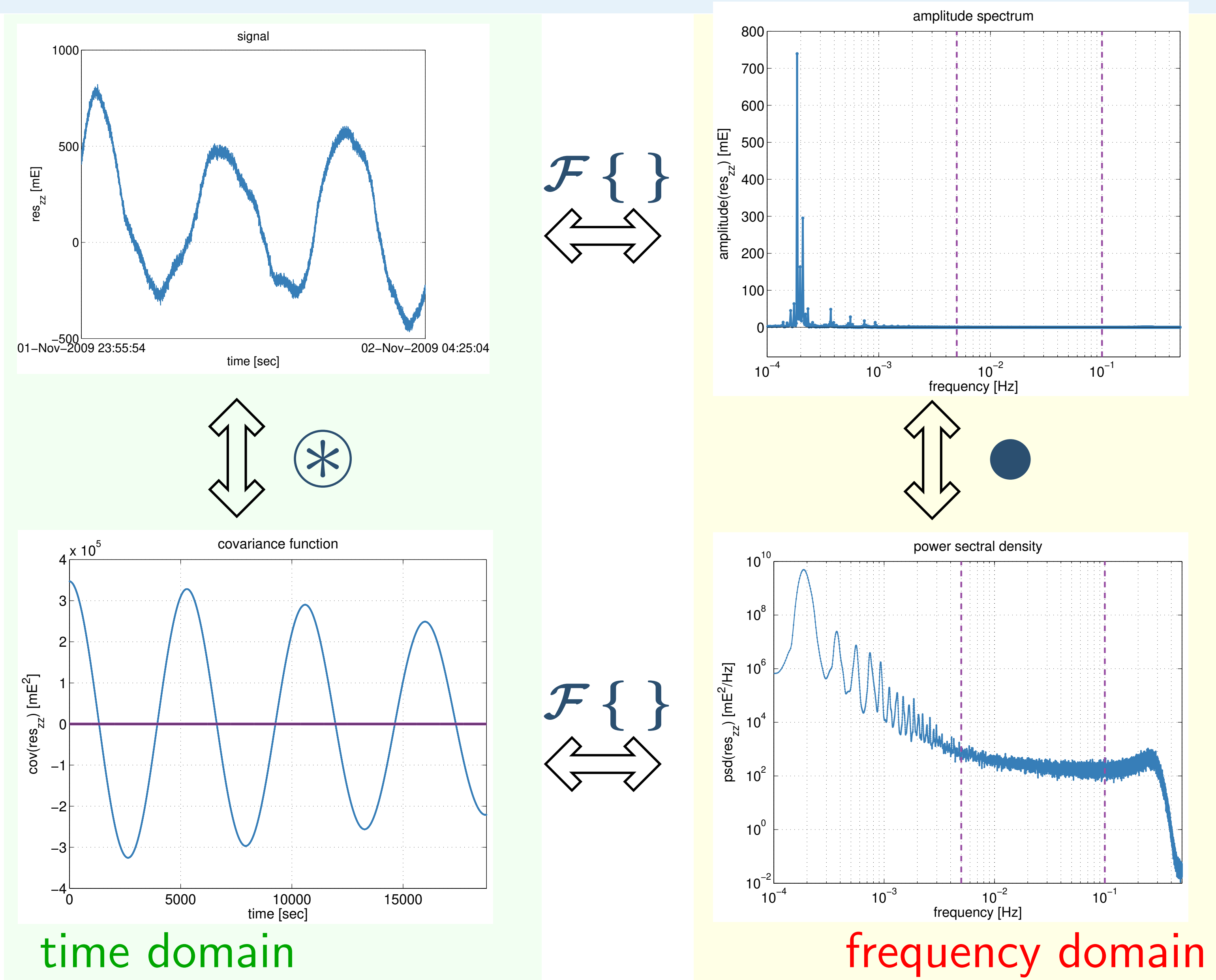






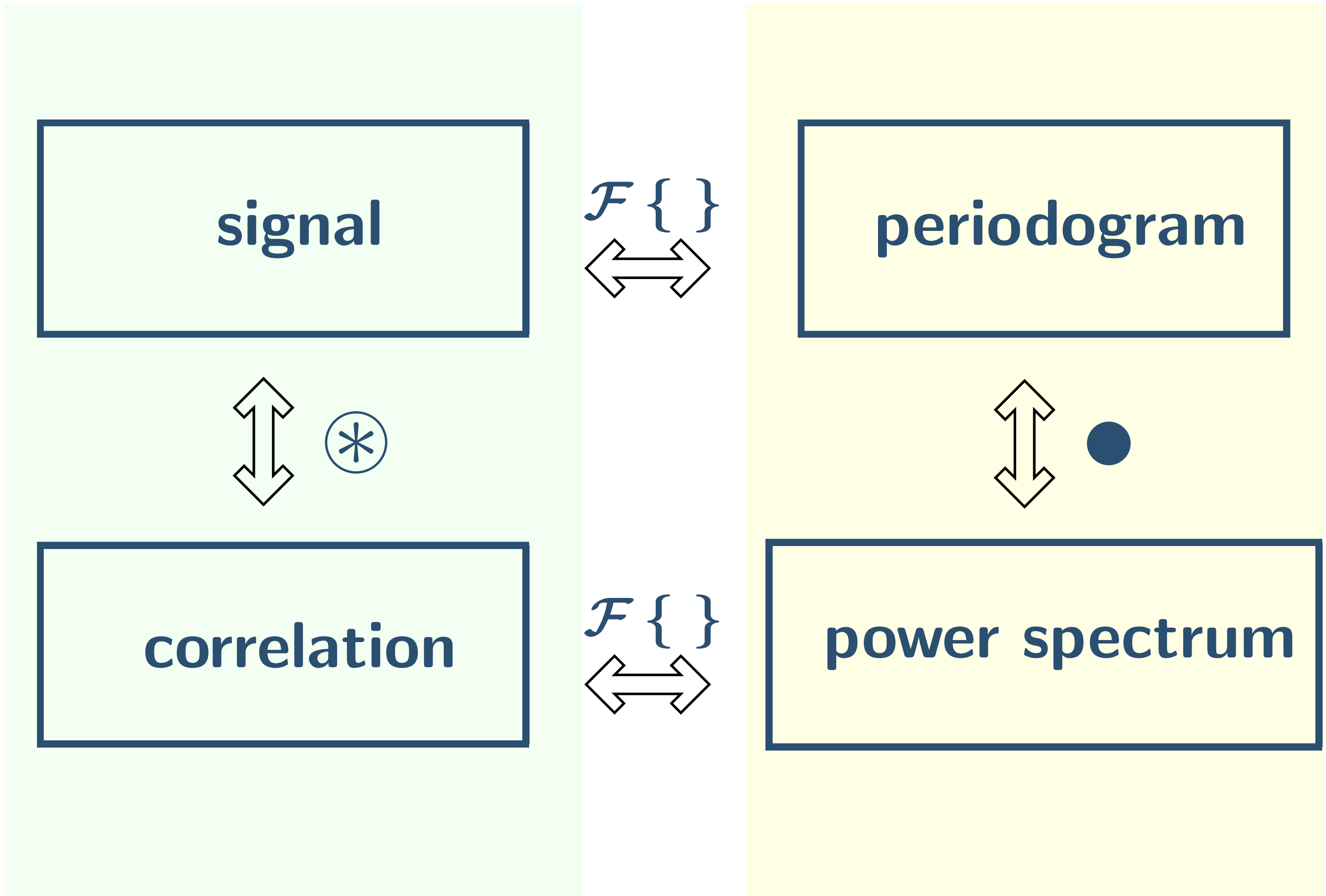
# Summary - Magic Square





# Magic Square






time domain

frequency domain

# Pre-whitening Filter

# Residual analysis: pre-whitening

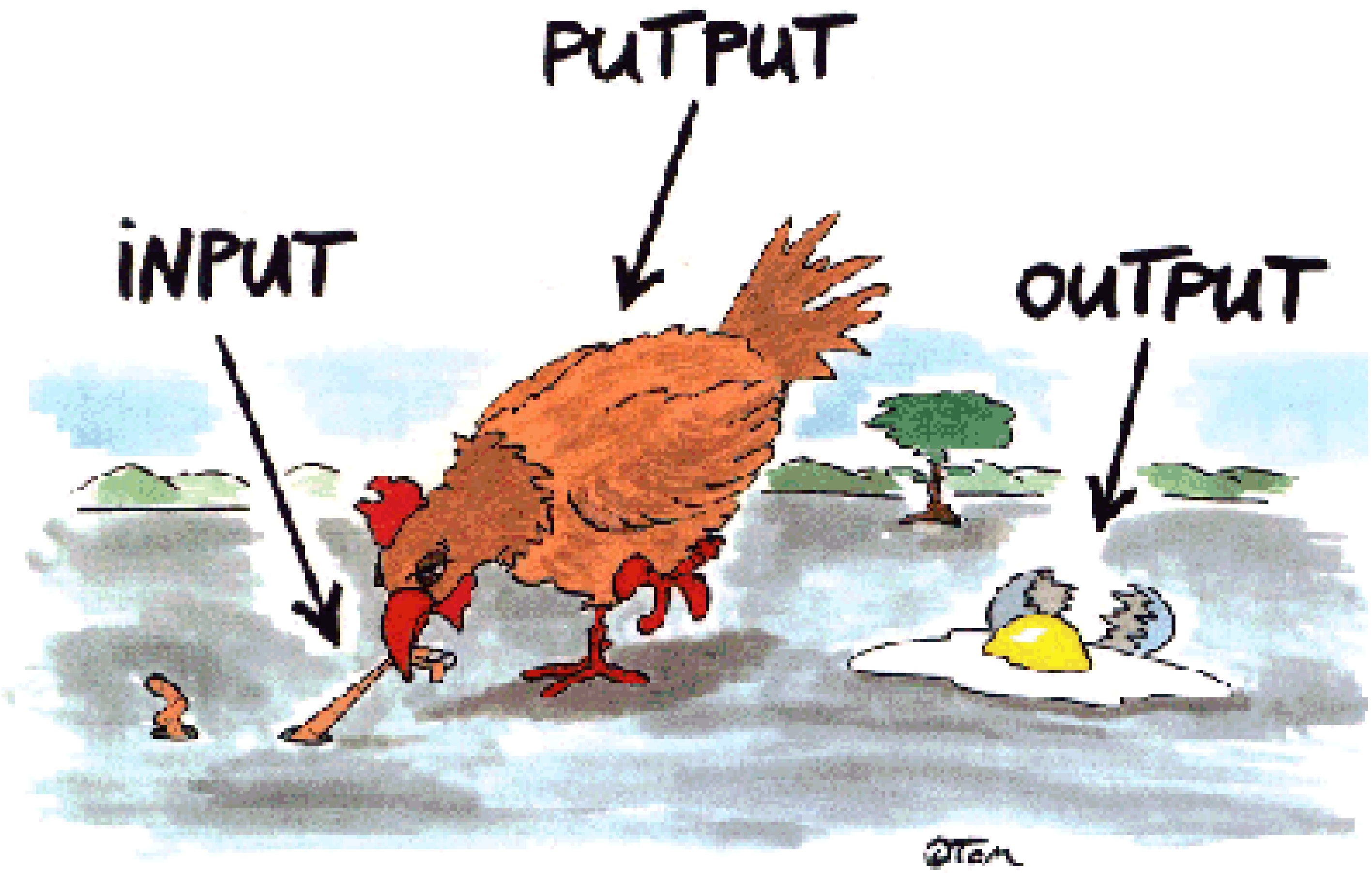


$$\begin{tikzcd}
\boxed{\ell} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow, "\circledast"] & \boxed{\mathcal{F}\{\ell\}} \arrow[d, Leftrightarrow, "\bullet"] \\
\boxed{\Sigma_{\ell\ell}} \arrow[r, Leftrightarrow] & \boxed{P_{\ell\ell}}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\begin{aligned}\bar{\ell} &= F\,\ell \\ &= f * \ell\end{aligned}} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow, "\circledast"] & \boxed{\begin{aligned}\mathcal{F}\{\bar{\ell}\} &= \mathcal{F}\{f * \ell\} \\ &= \mathcal{F}\{f\} \bullet \mathcal{F}\{\ell\}\end{aligned}} \arrow[d, Leftrightarrow, "\bullet"] \\
\boxed{\Sigma_{\bar{\ell}\bar{\ell}} = I} \arrow[r, Leftrightarrow] & \boxed{P_{ff} \bullet P_{\ell\ell} = \text{WGN}}
\end{tikzcd}$$

| | | |
|---|---|---|
| $f$ | ... | filter impulse responds |
| $\mathcal{F}\{f\}$ | ... | filter transfer function |
| $I$ | ... | identity matrix |
| WGN | ... | white Gaussian noise |

# Digital Filters


PUTPUT
INPUT
OUTPUT

# Filter



$$y(t) \Longleftarrow \boxed{\textbf{filter}} \Longleftarrow u(t)$$

# Filter

$$y(t) \Longleftarrow \boxed{\textbf{filter}} \Longleftarrow u(t)$$

- discrete $\quad \{y_n\} \Longleftarrow \boxed{\text{filter}} \Longleftarrow \{u_n\}$

# Filter

$$y(t) \Longleftarrow \boxed{\textbf{filter}} \Longleftarrow u(t)$$

- discrete $\{y_n\} \Longleftarrow \boxed{\text{filter}} \Longleftarrow \{u_n\}$
- linear $y_n = \sum_{k=-\infty}^{\infty} b_k u_{n-k} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} a_k y_{n-k}$

$$y(t) \Longleftarrow \boxed{\textbf{filter}} \Longleftarrow u(t)$$

- discrete $\{y_n\} \Longleftarrow \boxed{\text{filter}} \Longleftarrow \{u_n\}$
- linear $$y_n = \sum_{k=-\infty}^{\infty} b_k u_{n-k} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} a_k y_{n-k}$$
- finite $$y_n = \sum_{k=-p}^{p} b_k u_{n-k} - \sum_{\substack{k=-q \\ k\neq 0}}^{q} a_k y_{n-k}$$

$$y(t) \Longleftarrow \boxed{\textbf{filter}} \Longleftarrow u(t)$$

- discrete $\{y_n\} \Longleftarrow \boxed{\text{filter}} \Longleftarrow \{u_n\}$
- linear $$y_n = \sum_{k=-\infty}^{\infty} b_k u_{n-k} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} a_k y_{n-k}$$
- finite $$y_n = \sum_{k=-p}^{p} b_k u_{n-k} - \sum_{\substack{k=-q \\ k\neq 0}}^{q} a_k y_{n-k}$$
- causal $$y_n = \sum_{k=0}^{p} b_k u_{n-k} - \sum_{k=1}^{q} a_k y_{n-k}$$

$$y(t) \Longleftarrow \boxed{\text{filter}} \Longleftarrow u(t)$$

- discrete $\{y_n\} \Longleftarrow \boxed{\text{filter}} \Longleftarrow \{u_n\}$
- linear $y_n = \sum_{k=-\infty}^{\infty} b_k u_{n-k} - \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} a_k y_{n-k}$
- finite $y_n = \sum_{k=-p}^{p} b_k u_{n-k} - \sum_{\substack{k=-q \\ k \neq 0}}^{q} a_k y_{n-k}$
- causal $y_n = \sum_{k=0}^{p} b_k u_{n-k} - \sum_{k=1}^{q} a_k y_{n-k}$
- nonrecursive $y_n = \sum_{k=0}^{p} b_k u_{n-k}$

$$y(t) \Longleftarrow \boxed{\text{filter}} \Longleftarrow u(t)$$

- discrete $\{y_n\} \Longleftarrow \boxed{\text{filter}} \Longleftarrow \{u_n\}$
- linear $y_n = \sum_{k=-\infty}^{\infty} b_k u_{n-k} - \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} a_k y_{n-k}$
- finite $y_n = \sum_{k=-p}^{p} b_k u_{n-k} - \sum_{\substack{k=-q \\ k \neq 0}}^{q} a_k y_{n-k}$
- causal $y_n = \sum_{k=0}^{p} b_k u_{n-k} - \sum_{k=1}^{q} a_k y_{n-k}$
- nonrecursive $y_n = \sum_{k=0}^{p} b_k u_{n-k}$
- symmetric $y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

# Nonrecursive filter





$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

# Nonrecursive filter



$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

$c_{-2}$ $c_{-1}$ $c_0$ $c_1$ $c_2$

# Nonrecursive filter



$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

$$\begin{array}{ccccccccccc} & & & & & c_{-2} & c_{-1} & c_0 & c_1 & c_2 \\ \ldots & u_7 & u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & u_0 & u_{-1} & u_{-2} \end{array}$$

# Nonrecursive filter





$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

$$\begin{array}{cccccccccc} & & & & & c_{-2} & c_{-1} & c_0 & c_1 & c_2 \\ & & & & & * & * & * & * & * \\ \ldots & u_7 & u_6 & u_5 & u_4 & u_3 \quad u_2 & u_1 & u_0 & u_{-1} & u_{-2} \\ & & & & & + & + & + & + & + \end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{y_0}$$

$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

$$\begin{array}{cccccccccccc}
 & & & & & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & \\
 & & & & & * & * & * & * & * & \\
\ldots & u_7 & u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & u_0 & u_{-1} & u_{-2}
\end{array}$$

$$\underbrace{+ \quad + \quad + \quad + \quad +}_{y_1}$$

# Nonrecursive filter



$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

$$\begin{array}{ccccccccccc} & & & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & & \\ & & & * & * & * & * & * & & \\ \ldots \; u_7 & u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & u_0 & u_{-1} & u_{-2} \\ & & & + & + & + & + & + & & \end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{y_2}$$

$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

$$\begin{array}{cccccccccccc}
 & & & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & & & \\
 & & & * & * & * & * & * & & & \\
\dots & u_7 & u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & u_0 & u_{-1} & u_{-2} \\
 & & & + & + & + & + & + & & &
\end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{y_3}$$

$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

$$\begin{array}{cccccccccc}
 & & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & & & \\
 & & * & * & * & * & * & & & \\
\dots & u_7 & u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & u_0 & u_{-1} & u_{-2} \\
 & & + & + & + & + & + & & &
\end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{y_4}$$

# Nonrecursive filter



$$y_n = \sum_{k=-N}^{N} c_k u_{n-k}$$

$$\begin{array}{ccccccccccc}
 & & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & & & & \\
 & & * & * & * & * & * & & & & \\
\dots & u_7 & u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & u_0 & u_{-1} & u_{-2} \\
 & & + & + & + & + & + & & & & \\
 & & & & y_4 & & & & & &
\end{array}$$

$$\{y_k\} = \{c_k\} * \{u_k\}$$

# Discrete convolution

$$\boxed{\{y_k\} = \{c_k\} * \{u_k\}} \quad \overset{\mathcal{F}\{\ \}}{\Longleftrightarrow} \quad \boxed{Y(\nu)\Big|_{-\nu^N}^{\nu^N} = \ ?}$$

# Discrete convolution

$$\{y_k\} = \{c_k\} * \{u_k\} \quad \overset{\mathcal{F}\{\ \}}{\Longleftrightarrow} \quad Y(\nu)\Big|_{-\nu^N}^{\nu^N} = \ ?$$

| | | |
|---|---|---|
| $\{y_k\}$ | | $Y(\nu)\Big\|_{-\nu^N}^{\nu^N}$ |
| $\{c_k\}$ | $F(\nu) = \Delta t \sum_{k=-\infty}^{\infty} f_k e^{-i2\pi\nu k\Delta t}$ | $C(\nu)\Big\|_{-\nu^N}^{\nu^N}$ |
| $\{u_k\}$ | | $U(\nu)\Big\|_{-\nu^N}^{\nu^N}$ |

# Discrete convolution



$$\boxed{\{y_k\} = \{c_k\} * \{u_k\}} \quad \overset{\mathcal{F}\{\ \}}{\Longleftrightarrow} \quad \boxed{Y(\nu)\Big|_{-\nu^N}^{\nu^N} = \ ?}$$

$$\boxed{\{y_k\}} \qquad \boxed{Y(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

$$F(\nu) = \Delta t \sum_{k=-\infty}^{\infty} f_k e^{-i2\pi\nu k\Delta t}$$

$$\boxed{\{c_k\}} \qquad \boxed{C(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

$$\boxed{\{u_k\}} \qquad \boxed{U(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

$$\boxed{\{y_k\} = \{c_k\} * \{u_k\}} \quad \overset{\mathcal{F}\{\ \}}{\Longleftrightarrow} \quad \boxed{Y(\nu)\Big|_{-\nu^N}^{\nu^N} = \frac{1}{\Delta t} C(\nu) U(\nu)\Big|_{-\nu^N}^{\nu^N}}$$

# Transfer function for nonrecursive filters

$$Y(\nu) = \Delta t \sum_{k=-\infty}^{\infty} y_k e^{-i2\pi\nu k\Delta t}$$

$$\{y_k\} = \{c_k\} * \{u_k\} \qquad \rightleftharpoons \qquad Y(\nu)\Big|_{-\nu^N}^{\nu^N} = \frac{1}{\Delta t} C(\nu)U(\nu)\Big|_{-\nu^N}^{\nu^N}$$

$$y_k = \int_{-\nu^N}^{\nu^N} Y(\nu) e^{i2\pi\nu k\Delta t} d\nu$$

# Transfer function for nonrecursive filters



$$Y(\nu) = \Delta t \sum_{k=-\infty}^{\infty} y_k e^{-i2\pi\nu k\Delta t}$$

$$\{y_k\} = \{c_k\} * \{u_k\} \quad \rightleftharpoons \quad Y(\nu)\Big|_{-\nu^N}^{\nu^N} = \underbrace{\frac{1}{\Delta t} C(\nu)\, U(\nu)}_{:= H(\nu)}\Big|_{-\nu^N}^{\nu^N}$$

$$y_k = \int_{-\nu^N}^{\nu^N} Y(\nu) e^{i2\pi\nu k\Delta t} d\nu$$

transfer function

for nonrecursive filters

# Transfer function for nonrecursive filters



$$Y(\nu) = \Delta t \sum_{k=-\infty}^{\infty} y_k e^{-i2\pi\nu k\Delta t}$$

$$\{y_k\} = \{c_k\} * \{u_k\} \qquad \rightleftharpoons \qquad Y(\nu)\Big|_{-\nu^N}^{\nu^N} = \underbrace{\frac{1}{\Delta t} C(\nu)\, U(\nu)\Big|_{-\nu^N}^{\nu^N}}_{:= H(\nu)}$$

$$y_k = \int_{-\nu^N}^{\nu^N} Y(\nu) e^{i2\pi\nu k\Delta t} d\nu$$

transfer function

for nonrecursive filters

$$\boxed{\begin{aligned} H(\nu) &= \sum_{k=-\infty}^{\infty} c_k\, e^{-i2\pi\nu k\Delta t} \\ c_k &= \Delta t \int_{-\nu^N}^{\nu^N} H(\nu)\, e^{i2\pi\nu k\Delta t} d\nu \end{aligned}}$$

# Symmetric filters - examples





$\{c_k\}$ . . . **filter coefficients** $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

$\{c_k\}$ ... **filter coefficients** $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- **Moving average filter**
  $\frac{1}{3}\{\ 1 \quad 1 \quad 1\ \}$

# Symmetric filters - examples





$\{c_k\}$ ... **filter coefficients** $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- **Moving average filter**
- **Best fitting polynom filter**

# Symmetric filters - examples

$\{c_k\}$ ... **filter coefficients** $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- **Moving average filter**
- **Best fitting polynom filter**
  - **Quadratic polynom filter**

# Symmetric filters - examples



$\{c_k\}$ ... **filter coefficients** $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- **Moving average filter**
- **Best fitting polynom filter**
  - **Quadratic polynom filter**
    $\frac{1}{35}\{\,-3 \quad 12 \quad 17 \quad 12 \quad -3\,\}$ ... **5-points**

$\{c_k\}$ ... filter coefficients $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- **Moving average filter**
- **Best fitting polynom filter**
  - **Quadratic polynom filter**

    $\frac{1}{35}\{\,-3 \quad 12 \quad 17 \quad 12 \quad -3\,\}$ ... **5-points**

    $\frac{1}{21}\{\,-2 \quad 3 \quad 6 \quad 7 \quad 6 \quad 3 \quad -2\,\}$ ... **7-points**

# Symmetric filters - examples



$\{c_k\}$ ... filter coefficients $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- Moving average filter
- Best fitting polynom filter
  - Quadratic polynom filter
  - Modified quadratic polynom filter

# Symmetric filters - examples



$\{c_k\}$ ... filter coefficients $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- Moving average filter
- Best fitting polynom filter
  - Quadratic polynom filter
  - Modified quadratic polynom filter

    $\frac{1}{96}\{\, 7 \quad 24 \quad 34 \quad 24 \quad 7 \,\}$ ... 5-point-smoother

# Symmetric filters - examples





$\{c_k\}$ ... filter coefficients $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- Moving average filter
- Best fitting polynom filter
  - Quadratic polynom filter
  - Modified quadratic polynom filter

$\frac{1}{96}\{\ 7 \quad 24 \quad 34 \quad 24 \quad 7\ \}$ ... 5-point-smoother

$\frac{1}{980}\{-11\ 18\ 88\ 138\ 168\ 178\ 168\ 138\ 88\ 18\ -11\}$ ... 11-point-smoother

# Symmetric filters - examples



$\{c_k\}$ ... filter coefficients $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- Moving average filter
- Best fitting polynom filter
- Difference operator

  $\{-1 \quad 2 \ -1\ \}$ $\quad \ldots \quad$ $\Delta^2$-operator

# Symmetric filters - examples

$\{c_k\}$ . . . filter coefficients $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- Moving average filter
- Best fitting polynom filter
- Difference operator
- Interpolation filter

$\frac{1}{6}\{-1 \quad 4 \quad 0 \quad 4 \quad -1\}$ . . . cubic interpolator

# Symmetric filters - examples



$\{c_k\}$ . . . filter coefficients $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- Moving average filter
- Best fitting polynom filter
- Difference operator
- Interpolation filter

**Advantages of symmetric nonrecursive filters:**

# Symmetric filters - examples



$\{c_k\}$ ... filter coefficients $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- Moving average filter
- Best fitting polynom filter
- Difference operator
- Interpolation filter

**Advantages of symmetric nonrecursive filters:**

- flexible design

# Symmetric filters - examples



$\{c_k\}$ . . . **filter coefficients** $\quad y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$

- Moving average filter
- Best fitting polynom filter
- Difference operator
- Interpolation filter

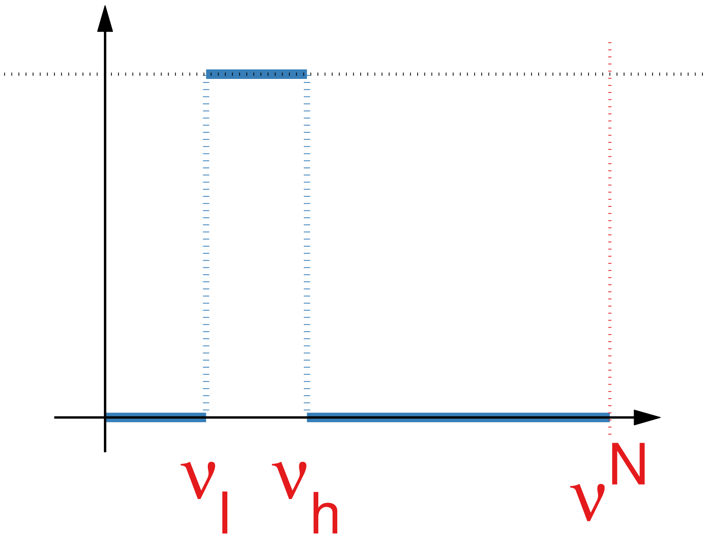
**Advantages of symmetric nonrecursive filters:**

- flexible design
- time/space conserving filter

# Ideal Bandpass Filter

# Ideal bandpass filter



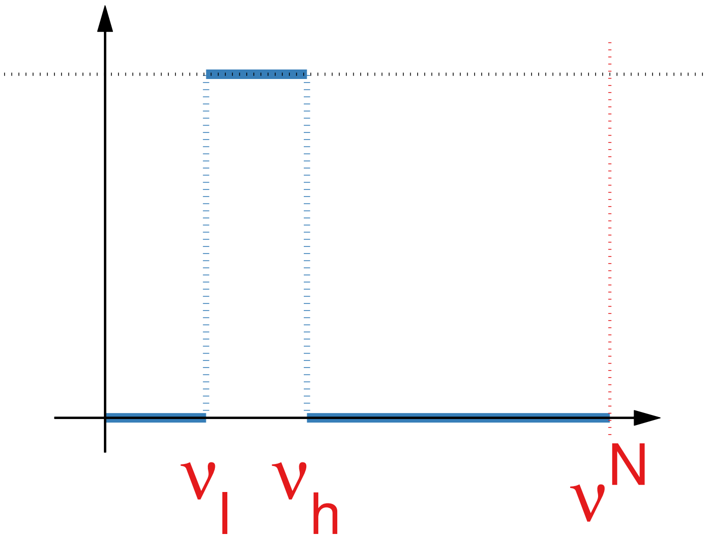



Symmetric nonrecursive filter:

$$y_n = \sum_{k=-\infty}^{\infty} c_{|k|} u_{n-k}$$

Ideal transfer function:

$$H_\circ^I(\nu) = \begin{cases} 0: & 0 \le |\nu| < \nu_l \\ 1: & \nu_l \le |\nu| < \nu_h \\ 0: & \nu_h \le |\nu| \le \nu^N \end{cases}$$




Symmetric nonrecursive filter:

$$y_n = \sum_{k=-\infty}^{\infty} c_{|k|} u_{n-k}$$

Ideal transfer function:

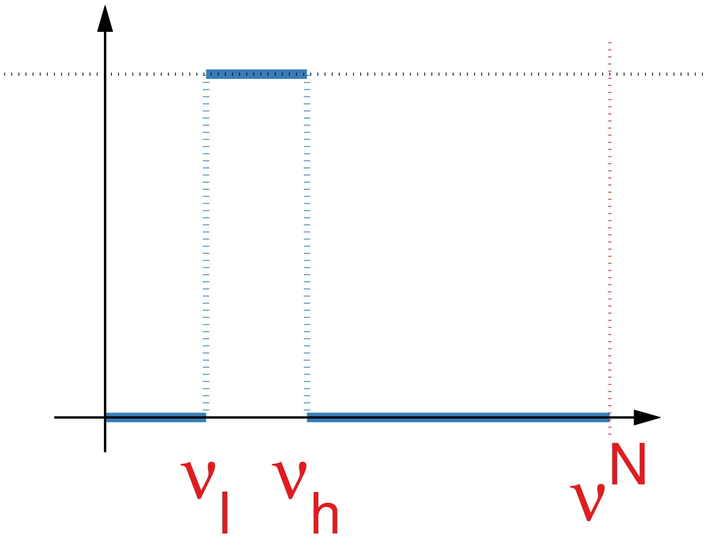
$$H_\circ^I(\nu) = \begin{cases} 0 : & 0 \leq |\nu| < \nu_l \\ 1 : & \nu_l \leq |\nu| < \nu_h \\ 0 : & \nu_h \leq |\nu| \leq \nu^N \end{cases}$$

Filter coefficients:

$$c_k^I = \Delta t \int_{-\nu^N}^{\nu^N} H^I(\nu)\, e^{i2\pi\nu k\Delta t} d\nu$$

# Ideal bandpass filter






Symmetric nonrecursive filter:

$$y_n = \sum_{k=-\infty}^{\infty} c_{|k|} u_{n-k}$$

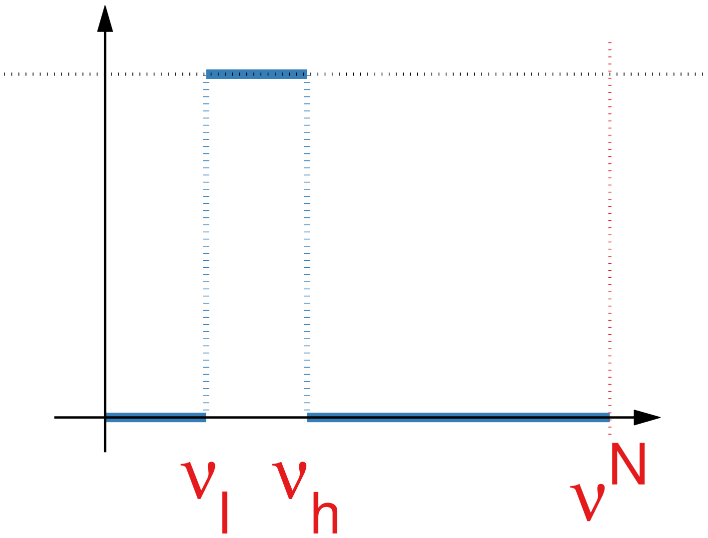
Ideal transfer function:

$$H_\circ^I(\nu) = \begin{cases} 0: & 0 \leq |\nu| < \nu_l \\ 1: & \nu_l \leq |\nu| < \nu_h \\ 0: & \nu_h \leq |\nu| \leq \nu^N \end{cases}$$

Filter coefficients:

$$c_k^I = \Delta t \int_{-\nu^N}^{\nu^N} H^I(\nu)\, e^{i2\pi\nu k \Delta t} d\nu$$

$$c_k^I = 2\,\Delta t \int_{\nu_l}^{\nu_h} \cos(2\pi\nu k \Delta t) d\nu$$




Symmetric nonrecursive filter:

$$y_n = \sum_{k=-\infty}^{\infty} c_{|k|} u_{n-k}$$

Ideal transfer function:

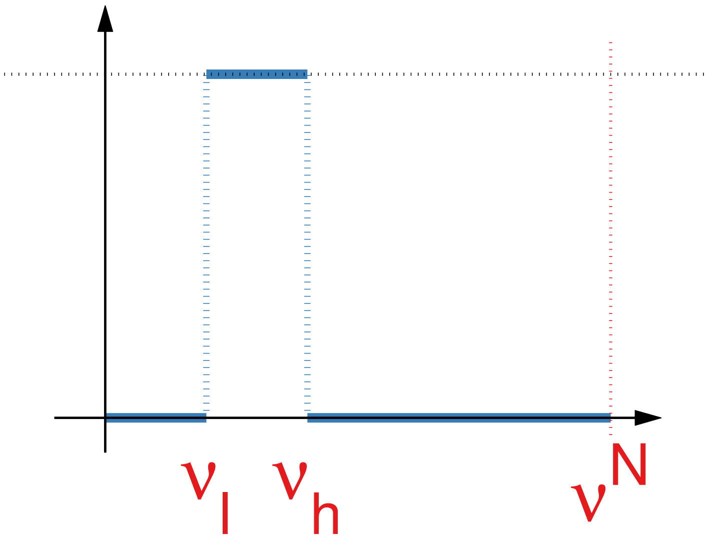
$$H_\circ^I(\nu) = \begin{cases} 0 : & 0 \le |\nu| < \nu_l \\ 1 : & \nu_l \le |\nu| < \nu_h \\ 0 : & \nu_h \le |\nu| \le \nu^N \end{cases}$$

Filter coefficients:

$$c_k^I = \Delta t \int_{-\nu^N}^{\nu^N} H^I(\nu)\; e^{i2\pi\nu k\Delta t} d\nu$$

$$c_k^I = 2\;\Delta t \int_{\nu_l}^{\nu_h} \cos(2\pi\nu k\Delta t) d\nu$$

$$c_k^I = \left.\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\right|_{\nu_l}^{\nu_h} \qquad k \in [-\infty, \infty]$$




Symmetric nonrecursive filter:

$$y_n = \sum_{k=-\infty}^{\infty} c_{|k|} u_{n-k}$$

Ideal transfer function:

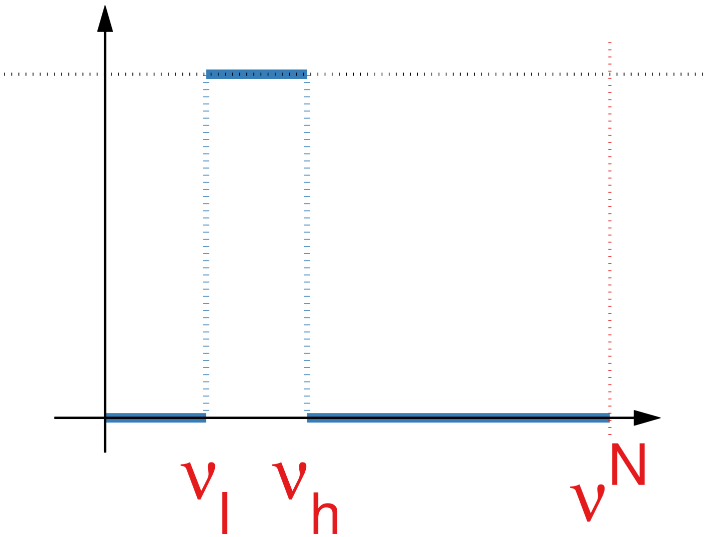
$$H_\circ^I(\nu) = \begin{cases} 0: & 0 \le |\nu| < \nu_l \\ 1: & \nu_l \le |\nu| < \nu_h \\ 0: & \nu_h \le |\nu| \le \nu^N \end{cases}$$

Filter coefficients:

$$c_k^I = \Delta t \int_{-\nu^N}^{\nu^N} H^I(\nu)\; e^{i2\pi\nu k\Delta t} d\nu$$

$$c_k^I = 2\;\Delta t \int_{\nu_l}^{\nu_h} \cos(2\pi\nu k\Delta t) d\nu$$

$$c_k^I = \left.\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\right|_{\nu_l}^{\nu_h} \qquad k \in [-\infty, \infty]$$




Symmetric nonrecursive filter:

$$y_n = \sum_{k=-\infty}^{\infty} c_{|k|} u_{n-k}$$

Ideal transfer function:

$$H_\circ^I(\nu) = \begin{cases} 0 : & 0 \le |\nu| < \nu_l \\ 1 : & \nu_l \le |\nu| < \nu_h \\ 0 : & \nu_h \le |\nu| \le \nu^N \end{cases}$$

Filter coefficients:

$$\boxed{c_k^I = \frac{\sin(2\pi\nu k\Delta t)}{\pi k}\bigg|_{\nu_l}^{\nu_h} \qquad k \in [-\infty, \infty]}$$

## Truncated symmetric nonrecursive filter:

$$y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$$

## Truncated symmetric nonrecursive filter:

$$y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$$

## Filter coefficients:

$$c_k^I = \frac{\sin(2\pi\nu k\Delta t)}{\pi k}\Bigg|_{\nu_l}^{\nu_h} \qquad k \in [-N, N]$$

## Truncated symmetric nonrecursive filter:

$$y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$$

## Filter coefficients:

$$c_k^I = \frac{\sin(2\pi\nu k\Delta t)}{\pi k}\Bigg|_{\nu_l}^{\nu_h} \qquad k \in [-N, N]$$

## Transfer function:

$$H^T(\nu) = \sum_{k=-N}^{N} c_k e^{-i2\pi\nu k\Delta t}$$

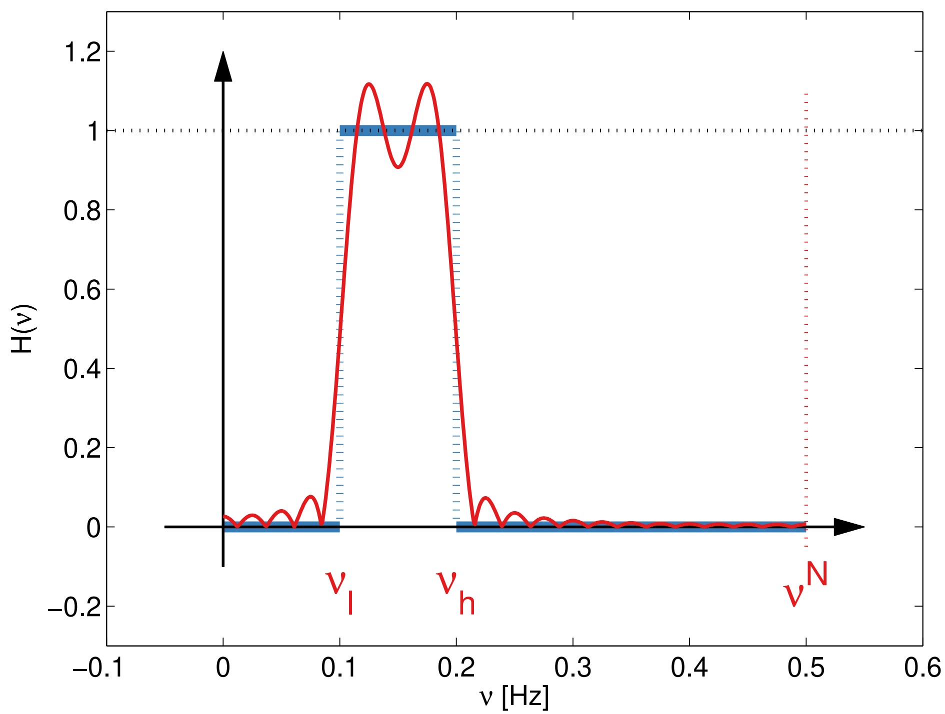
# Truncated ideal bandpass filter



## Truncated symmetric nonrecursive filter:

$$y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$$

**N = 20**




## Filter coefficients:

$$c_k^I = \left. \frac{\sin(2\pi\nu k\Delta t)}{\pi k} \right|_{\nu_l}^{\nu_h}$$

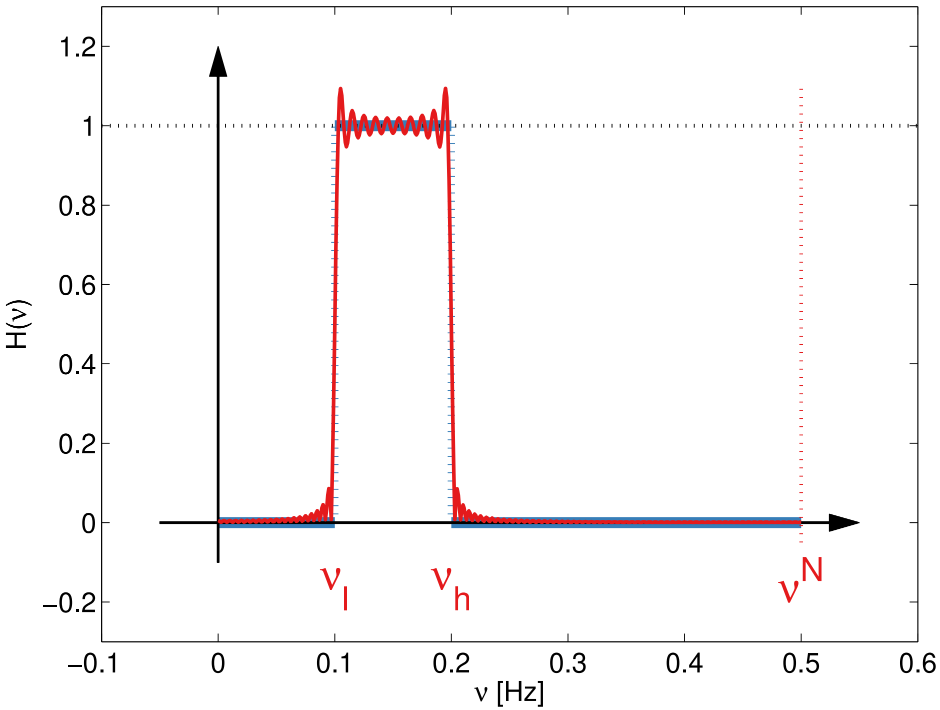
## Transfer function:

$$H^T(\nu) = \sum_{k=-N}^{N} c_k e^{-i2\pi\nu k\Delta t}$$

## Truncated symmetric nonrecursive filter:

$$y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$$

N = 100




## Filter coefficients:

$$c_k^I = \frac{\sin(2\pi\nu k\Delta t)}{\pi k}\bigg|_{\nu_l}^{\nu_h}$$

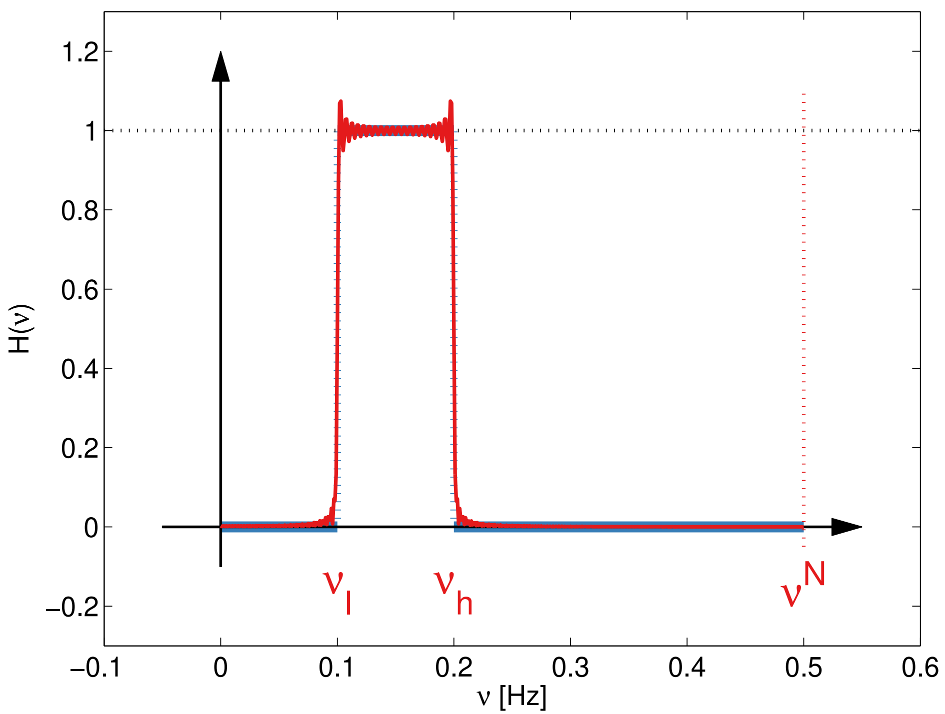
## Transfer function:

$$H^T(\nu) = \sum_{k=-N}^{N} c_k e^{-i2\pi\nu k\Delta t}$$

## Truncated symmetric nonrecursive filter:

$$y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$$

**N = 200**




## Filter coefficients:

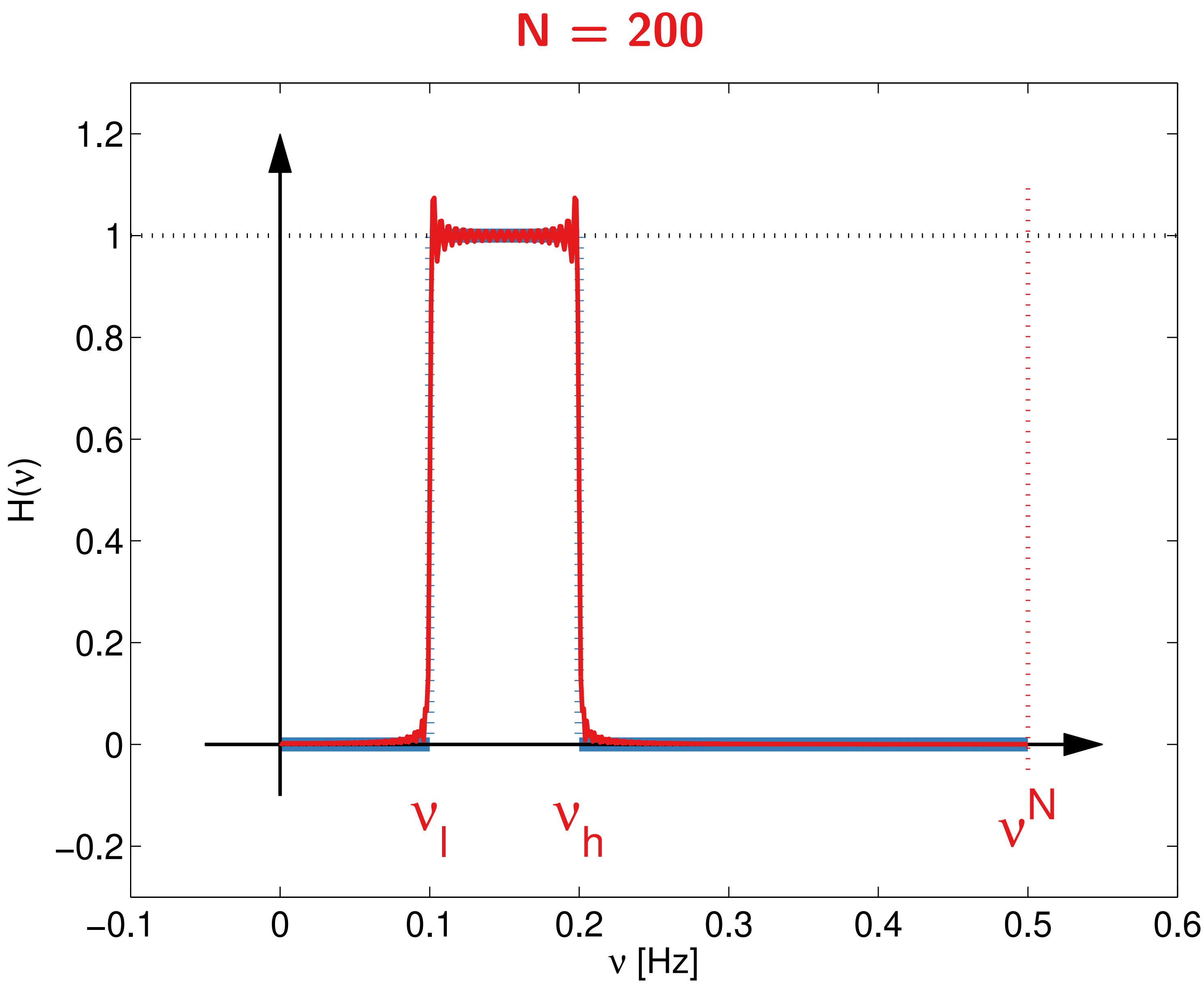
$$c_k^I = \frac{\sin(2\pi\nu k\Delta t)}{\pi k}\Bigg|_{\nu_l}^{\nu_h}$$

## Transfer function:

$$H^T(\nu) = \sum_{k=-N}^{N} c_k e^{-i2\pi\nu k\Delta t}$$

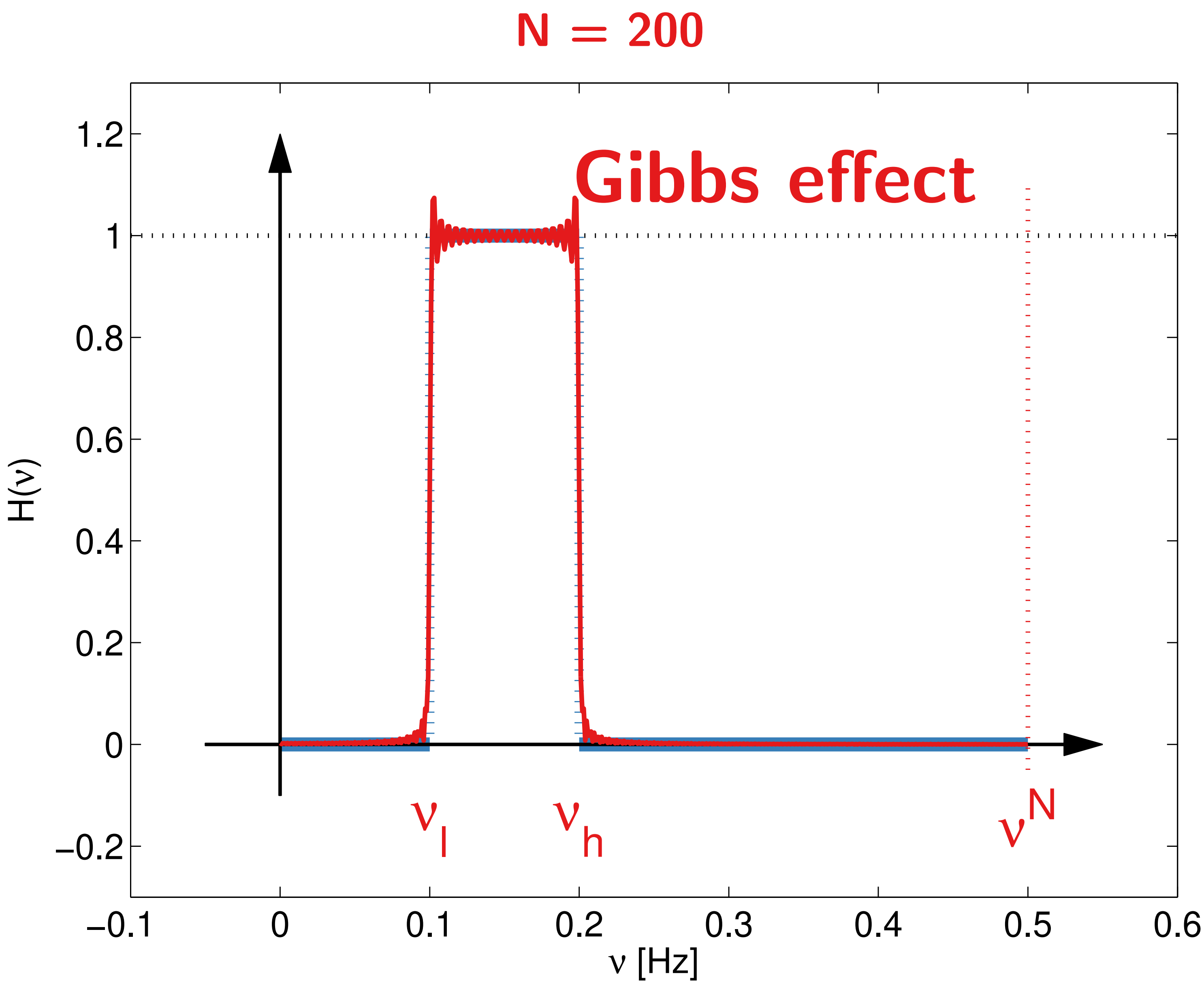
# Truncated ideal bandpass filter





# Truncated ideal bandpass filter





**Gipps effect:** ripples in the frequency of the first neglected term

**Gipps effect:** ripples in the frequency of the first neglected term

$$2\pi\nu N\Delta t = 2\pi$$

# Lanczos smoothing - sigma factors

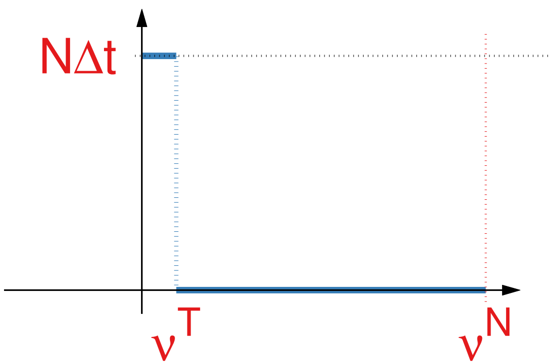


**Gipps effect:** ripples in the frequency of the first neglected term

$$2\pi\nu N\Delta t = 2\pi \quad \rightarrow \quad \boxed{\nu^T = \frac{1}{N\Delta t}}$$

**Gipps effect:** ripples in the frequency of the first neglected term

$$2\pi\nu N\Delta t = 2\pi \quad \rightarrow \quad \boxed{\nu^T = \frac{1}{N\Delta t}}$$

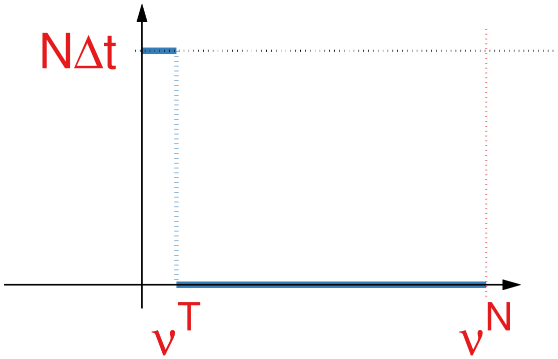



Window function - frequency domain:

$$W^L(\nu) = \begin{cases} N\Delta t : & 0 \leq |\nu| < \frac{1}{2}\nu^T \\ 0 : & \frac{1}{2}\nu^T \leq |\nu| < \nu^N \end{cases}$$

**Gipps effect:** ripples in the frequency of the first neglected term

$$2\pi\nu N\Delta t = 2\pi \quad \rightarrow \quad \boxed{\nu^T = \frac{1}{N\Delta t}}$$




Window function - frequency domain:

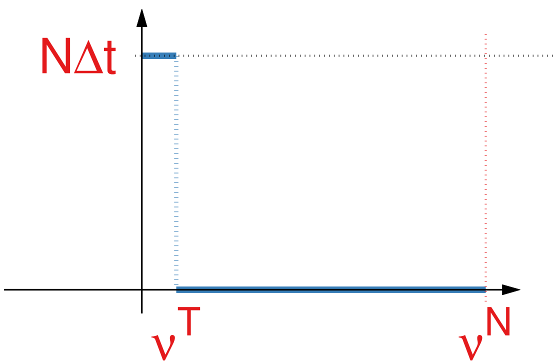
$$W^L(\nu) = \begin{cases} N\Delta t : & 0 \leq |\nu| < \frac{1}{2}\nu^T \\ 0 : & \frac{1}{2}\nu^T \leq |\nu| < \nu^N \end{cases}$$

**Convolution in the frequency domain:**

$$H^S(\nu) = H^T(\nu) * W^L(\nu)$$

**Gipps effect:** ripples in the frequency of the first neglected term

$$2\pi\nu N\Delta t = 2\pi \quad \rightarrow \quad \boxed{\nu^T = \frac{1}{N\Delta t}}$$




Window function - frequency domain:

$$W^L(\nu) = \begin{cases} N\Delta t : & 0 \leq |\nu| < \frac{1}{2}\nu^T \\ 0 : & \frac{1}{2}\nu^T \leq |\nu| < \nu^N \end{cases}$$

**Convolution in the frequency domain:**

$$H^S(\nu) = H^T(\nu) * W^L(\nu)$$

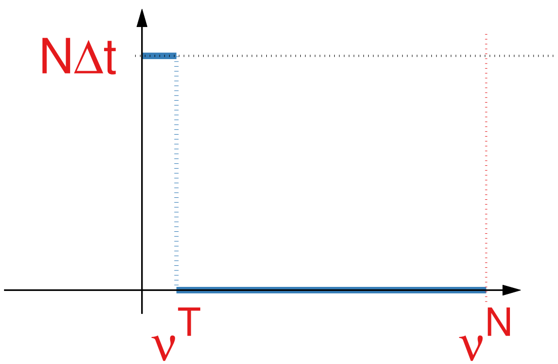
$$H^S(\nu) = N\Delta t \int_{\nu-\frac{1}{2}\nu^T}^{\nu+\frac{1}{2}\nu^T} H^T(\nu)\, d\nu$$

# Lanczos smoothing - sigma factors



**Gipps effect:** ripples in the frequency of the first neglected term

$$2\pi\nu N\Delta t = 2\pi \quad \rightarrow \quad \boxed{\nu^T = \frac{1}{N\Delta t}}$$




Window function - frequency domain:

$$W^L(\nu) = \begin{cases} N\Delta t : & 0 \leq |\nu| < \frac{1}{2}\nu^T \\ 0 : & \frac{1}{2}\nu^T \leq |\nu| < \nu^N \end{cases}$$

**Convolution in the frequency domain:**

$$H^S(\nu) = H^T(\nu) * W^L(\nu)$$

$$H^S(\nu) = \sum_{k=-N}^{N} \frac{\sin(2\pi\nu k\Delta t)}{\pi k}\bigg|_{\nu_l}^{\nu_k} \frac{\sin(\pi k/N)}{\pi k/N}\, e^{-i2\pi\nu k\Delta t}$$

$$H^S(\nu) = \sum_{k=-N}^{N} \left.\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\right|_{\nu_l}^{\nu_k} \frac{\sin(\pi k/N)}{\pi k/N} e^{-i2\pi\nu k\Delta t}$$

# Lanczos smoothing - sigma factors





$$H^S(\nu) = \sum_{k=-N}^{N} \underbrace{\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\bigg|_{\nu_l}^{\nu_k}}_{c_k^I} \frac{\sin(\pi k/N)}{\pi k/N} e^{-i2\pi\nu k\Delta t}$$

$$H^S(\nu) = \sum_{k=-N}^{N} \underbrace{\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\Big|_{\nu_l}^{\nu_k}}_{c_k^I} \underbrace{\frac{\sin(\pi k/N)}{\pi k/N}}_{\sigma(k,N)} e^{-i2\pi\nu k\Delta t}$$

$$H^S(\nu) = \sum_{k=-N}^{N} \underbrace{\underbrace{\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\bigg|_{\nu_l}^{\nu_k}}_{c_k^I} \underbrace{\frac{\sin(\pi k/N)}{\pi k/N}}_{\sigma(k,N)}}_{:= c_k^S} e^{-i2\pi\nu k\Delta t}$$

# Lanczos smoothing - sigma factors



$$H^S(\nu) = \sum_{k=-N}^{N} \underbrace{\underbrace{\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\Bigg|_{\nu_l}^{\nu_k}}_{c_k^I} \underbrace{\frac{\sin(\pi k/N)}{\pi k/N}}_{\sigma(k,N)}}_{:= c_k^S} e^{-i2\pi\nu k\Delta t}$$

$$H^S(\nu) = \sum_{k=-N}^{N} c_k^S \, e^{-i2\pi\nu k\Delta t}$$

# Lanczos smoothing - sigma factors



$$H^S(\nu) = \sum_{k=-N}^{N} \underbrace{\underbrace{\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\Bigg|_{\nu_l}^{\nu_k}}_{c_k^I} \; \underbrace{\frac{\sin(\pi k/N)}{\pi k/N}}_{\sigma(k,N)}}_{:= c_k^S} \; e^{-i2\pi\nu k\Delta t}$$

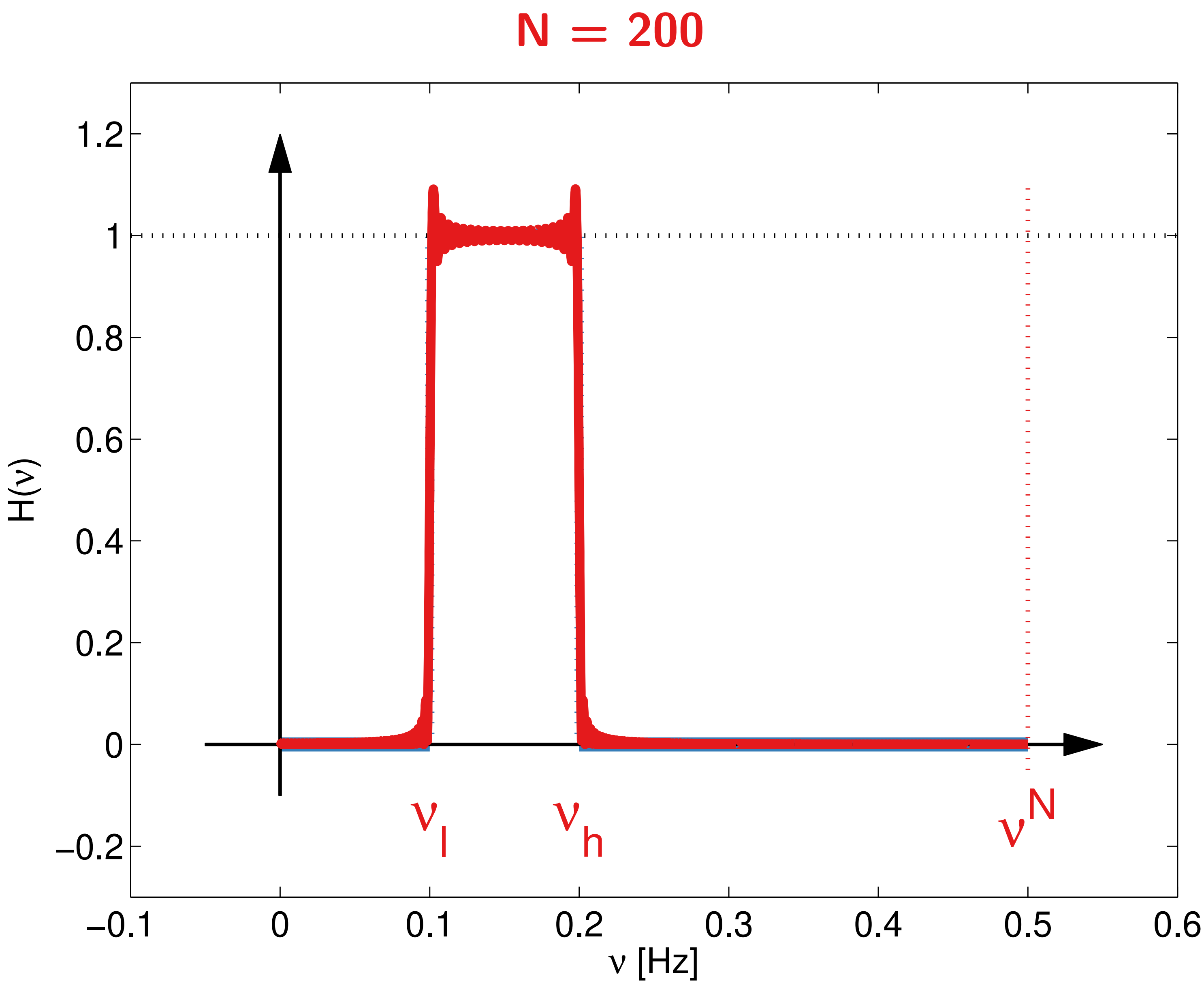
$$H^S(\nu) = \sum_{k=-N}^{N} c_k^S \; e^{-i2\pi\nu k\Delta t}$$

$$H^S(\nu) = \sum_{k=-N}^{N} c_k^S \; \cos(2\pi\nu k\Delta t)$$

# Lanczos smoothing - sigma factors



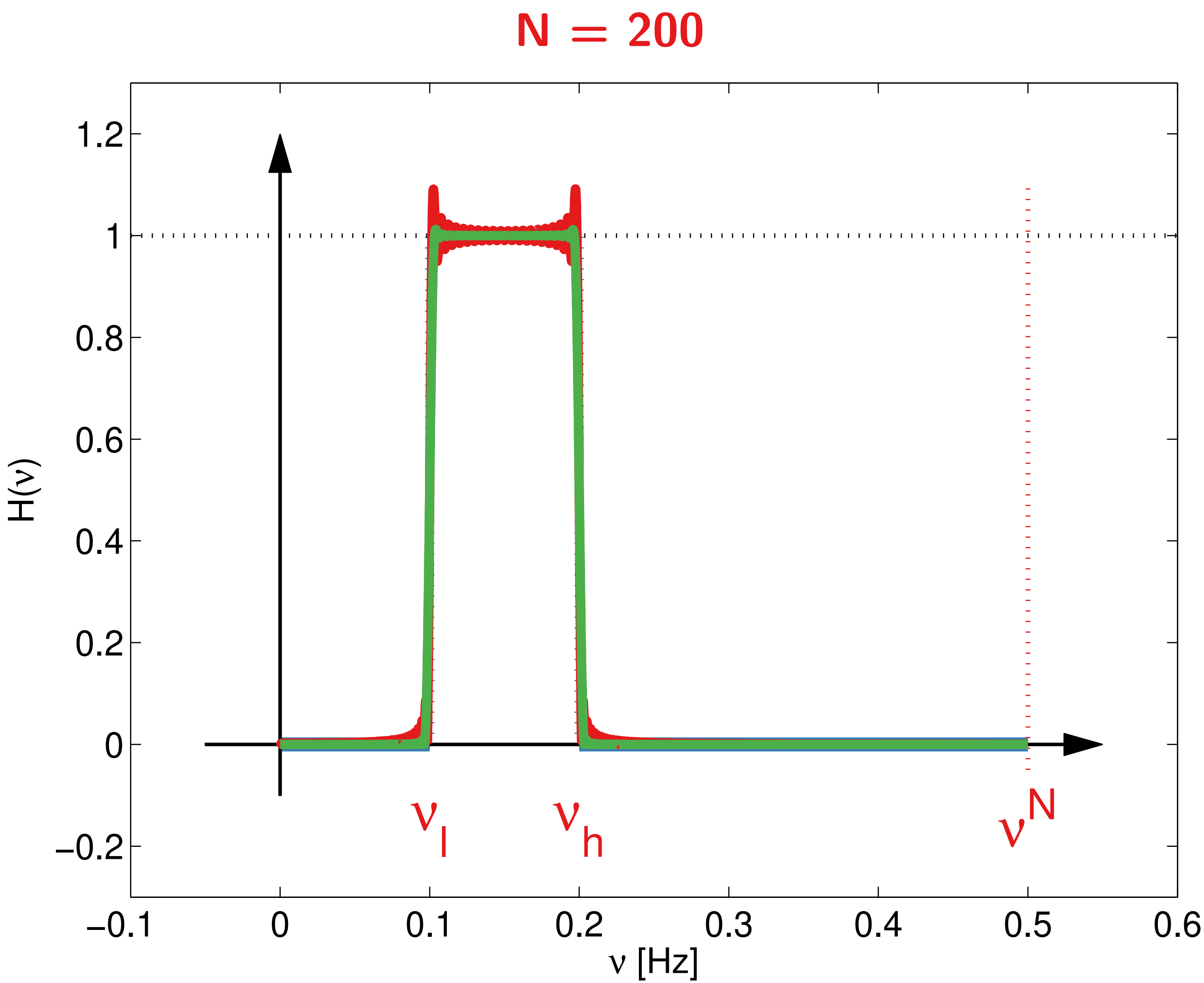




# Lanczos smoothing - sigma factors

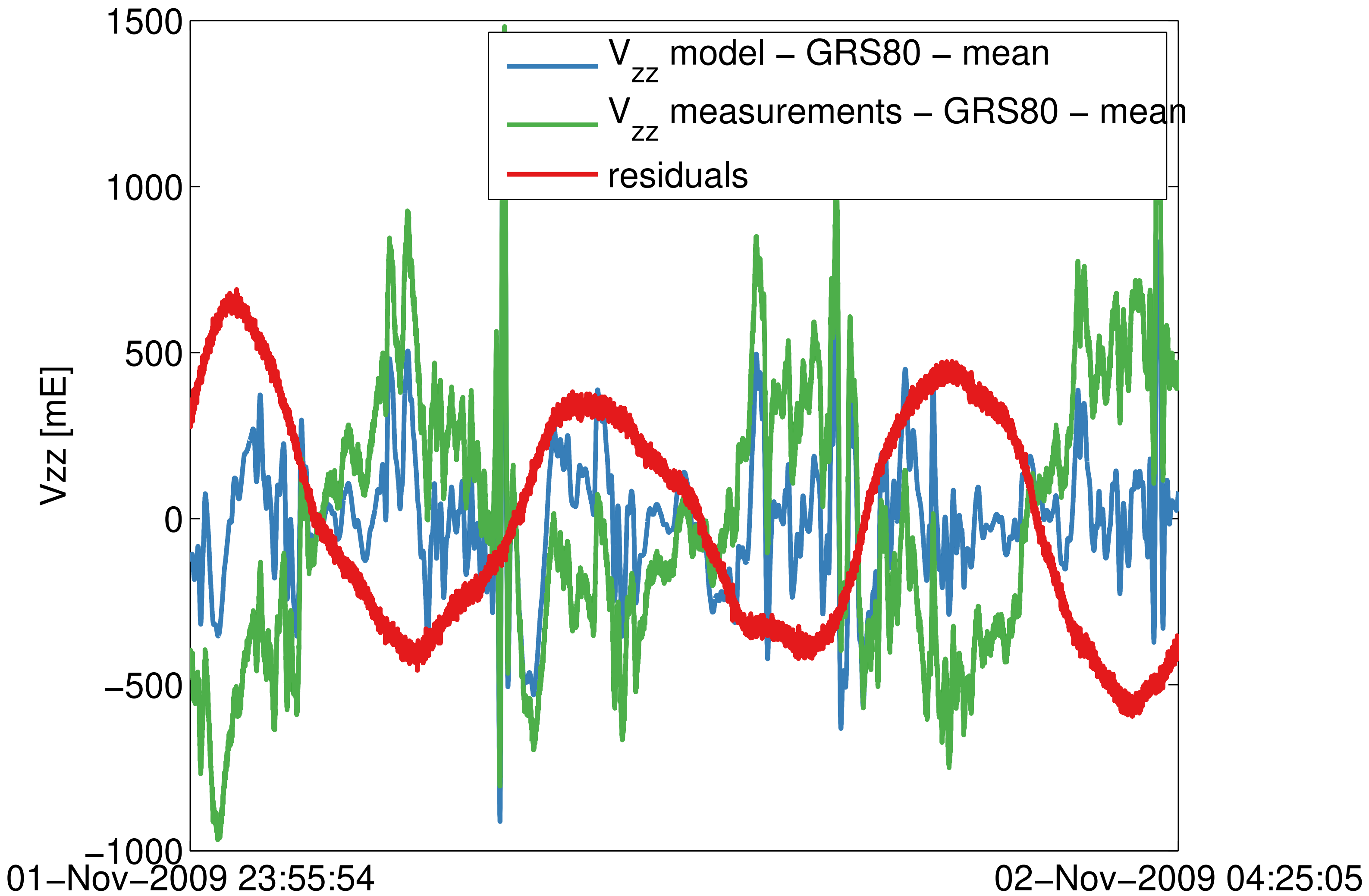






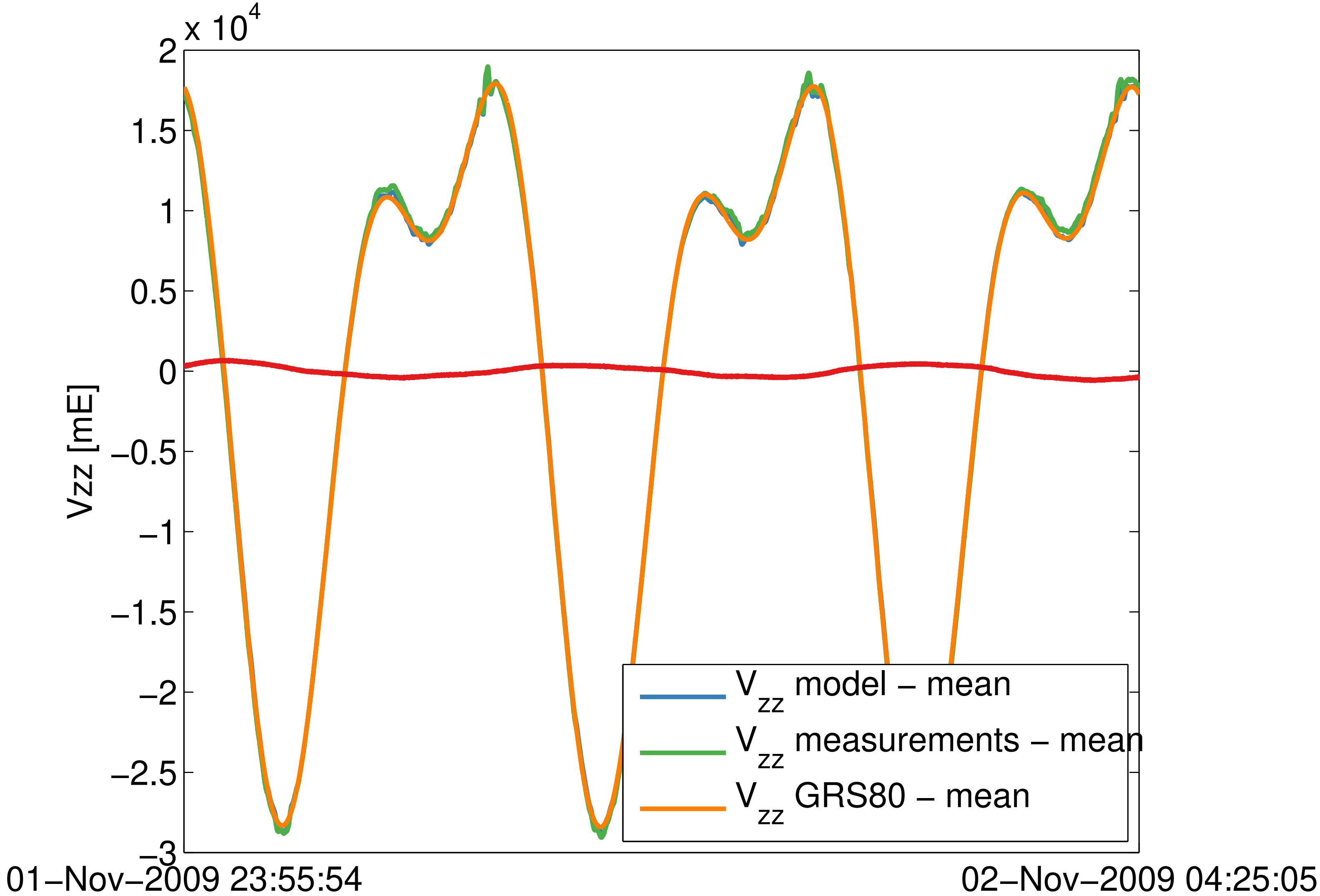
# GOCE - EGG-NOM_2 $V_{zz}$ signal





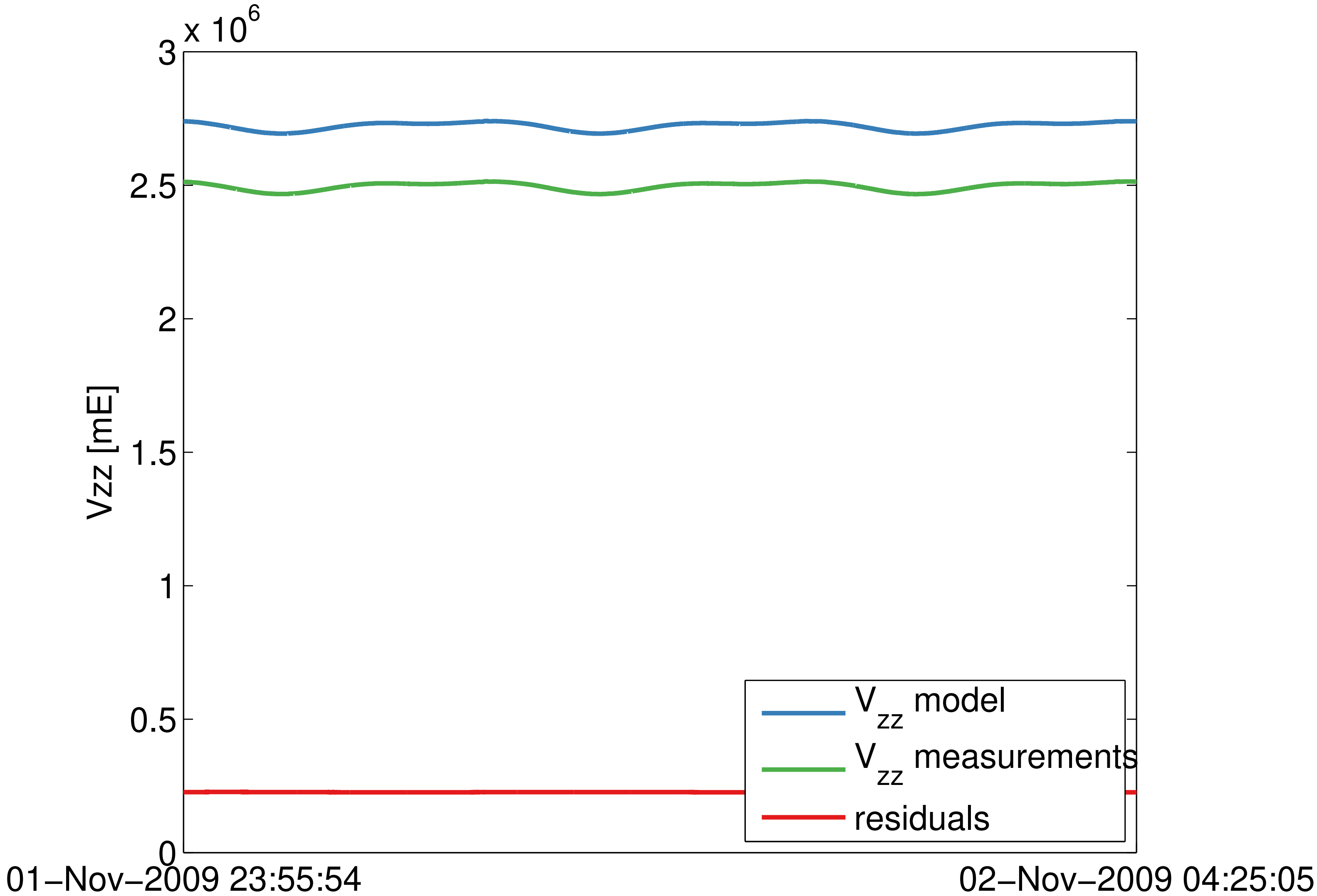
# GOCE - EGG-NOM_2 $V_{zz}$ signal





# GOCE - EGG-NOM_2 $V_{zz}$ signal





## GOCE EGG-NOM_2 $V_{zz}$ signal:

$\{\boldsymbol{u_k}\}$ residuals with a large bias

**GOCE EGG-NOM_2 $V_{zz}$ signal:**

$\{u_k\}$ residuals with a large bias

**requirement:** $\{y_k\}$ ... zero mean

**GOCE EGG-NOM_2 $V_{zz}$ signal:**

$\{u_k\}$ residuals with a large bias

**requirement:** $\{y_k\}$ ... zero mean

$$H^S(\nu)\big|_{\nu=0} \stackrel{!}{=} 0$$

$$H^S(\nu)\big|_{\nu=0} = \sum_{k=-N}^{N} c^S_{|k|} \stackrel{!}{=} 0$$ Parseval-theorem

**GOCE EGG-NOM_2 $V_{zz}$ signal:**

$\{u_k\}$ residuals with a large bias

**requirement:** $\{y_k\}$ ... zero mean

$$H^S(\nu)\Big|_{\nu=0} \stackrel{!}{=} 0$$

$$H^S(\nu)\Big|_{\nu=0} = \sum_{k=-N}^{N} c^S_{|k|} \stackrel{!}{=} 0 \qquad \text{Parseval-theorem}$$

**condition:** $$\sum_{k=-N+1}^{N-1} c^S_{|k|} + 2\,c_N = 0$$

**GOCE EGG-NOM_2 $V_{zz}$ signal:**

$\{u_k\}$ residuals with a large bias

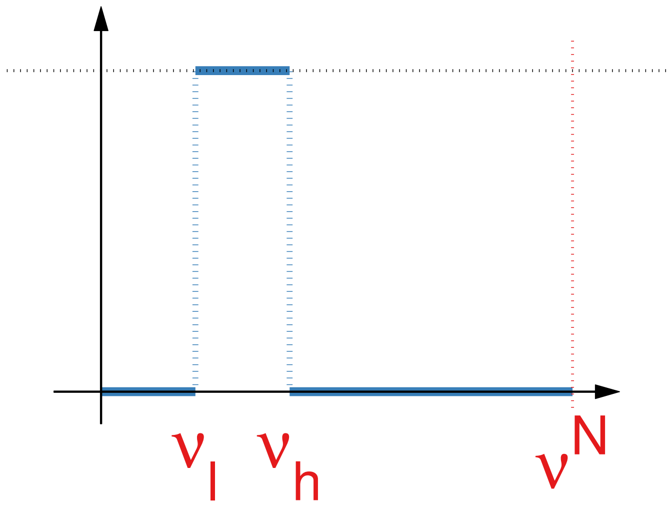
**requirement:** $\{y_k\}$ ... zero mean

$$H^S(\nu)\big|_{\nu=0} \overset{!}{=} 0$$

$$H^S(\nu)\big|_{\nu=0} = \sum_{k=-N}^{N} c^S_{|k|} \overset{!}{=} 0 \qquad \text{Parseval-theorem}$$

**condition:** $$\sum_{k=-N+1}^{N-1} c^S_{|k|} + 2\, c_N = 0$$

$$\boxed{\begin{aligned} c_N &= -\frac{1}{2} \sum_{k=-N+1}^{N-1} c^S_{|k|} \\ c_k &= c^S_k \qquad\qquad k \in [-N+1, N-1] \end{aligned}}$$




Symmetric nonrecursive filter:

$$y_n = \sum_{k=-N}^{N} c_{|k|} u_{n-k}$$

Ideal transfer function:

$$H_\circ^I(\nu) = \begin{cases} 0: & 0 \le \nu < \nu_l \\ 1: & \nu_l \le \nu < \nu_h \\ 0: & \nu_h \le \nu \le \nu^N \end{cases}$$

Filter coefficients:

$$c_{|k|} = \begin{cases} 2\Delta t(\nu_h - \nu_l) & : \quad k = 0 \\ \left.\frac{\sin(2\pi\nu k\Delta t)}{\pi k}\right|_{\nu_l}^{\nu_h} \frac{\sin(\pi k/N)}{\pi k/N} & : \quad k = 1, \ldots, N-1 \\ -\frac{1}{2} \sum_{j=-N+1}^{N-1} c_{|j|} & : \quad k = N \end{cases}$$

Filter transfer function:

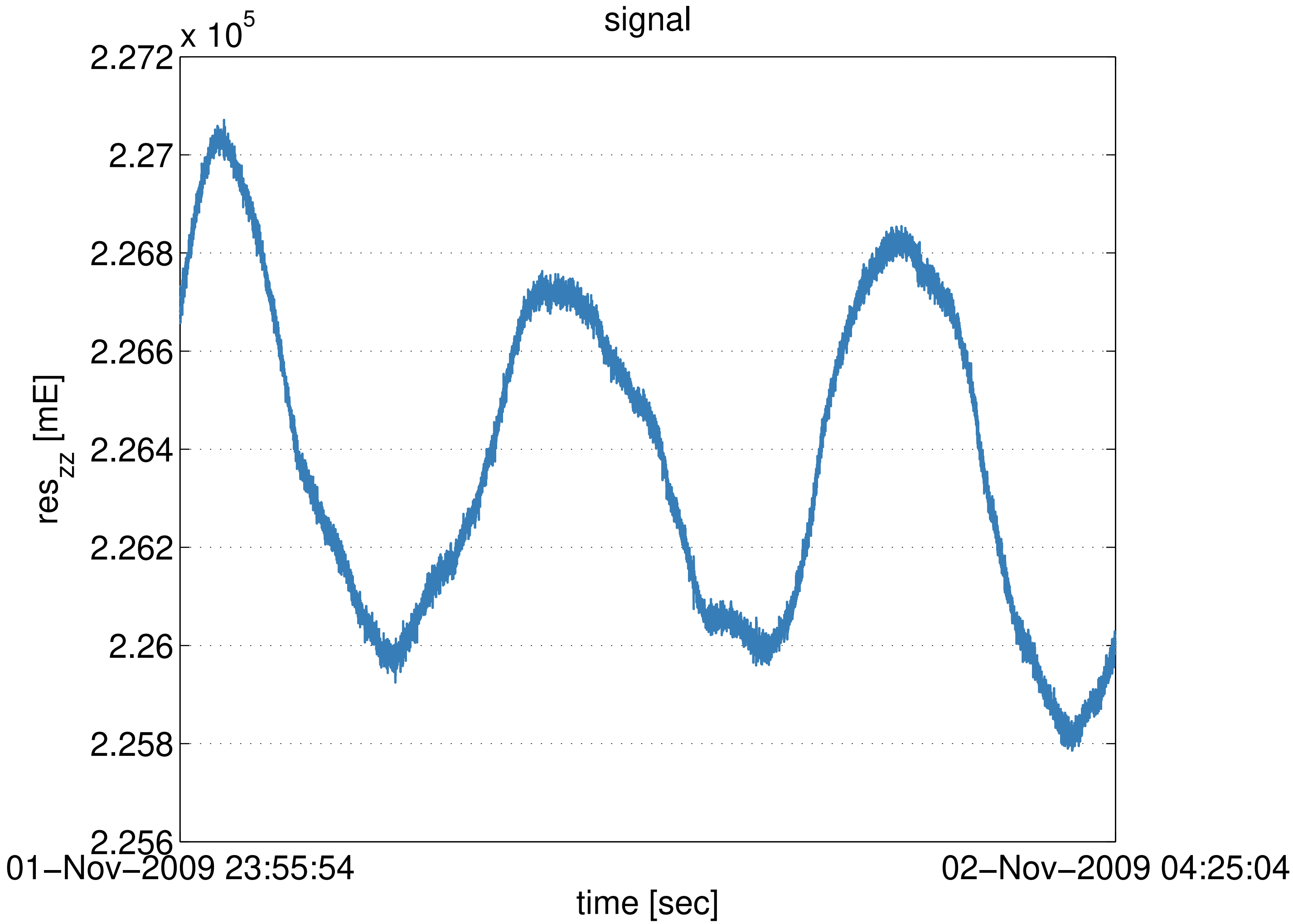
$$H^F(\nu) = \sum_{k=-N}^{N} c_{|k|} e^{-i2\pi\nu k\Delta t} \Bigg|_{\nu^N}^{-\nu^N}$$

# GOCE Filter

# GOCE - filter 9299



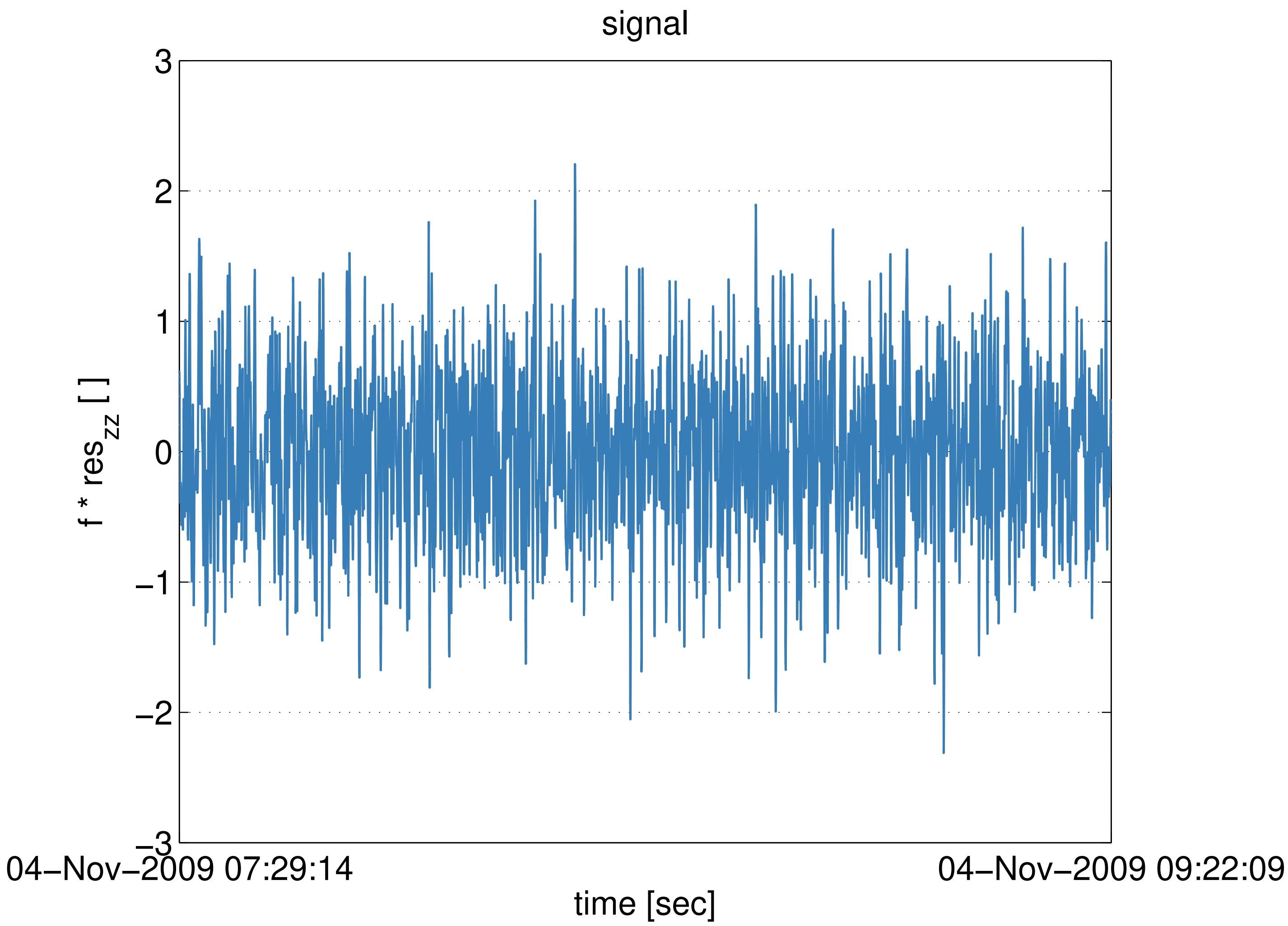




# GOCE - filter 9299

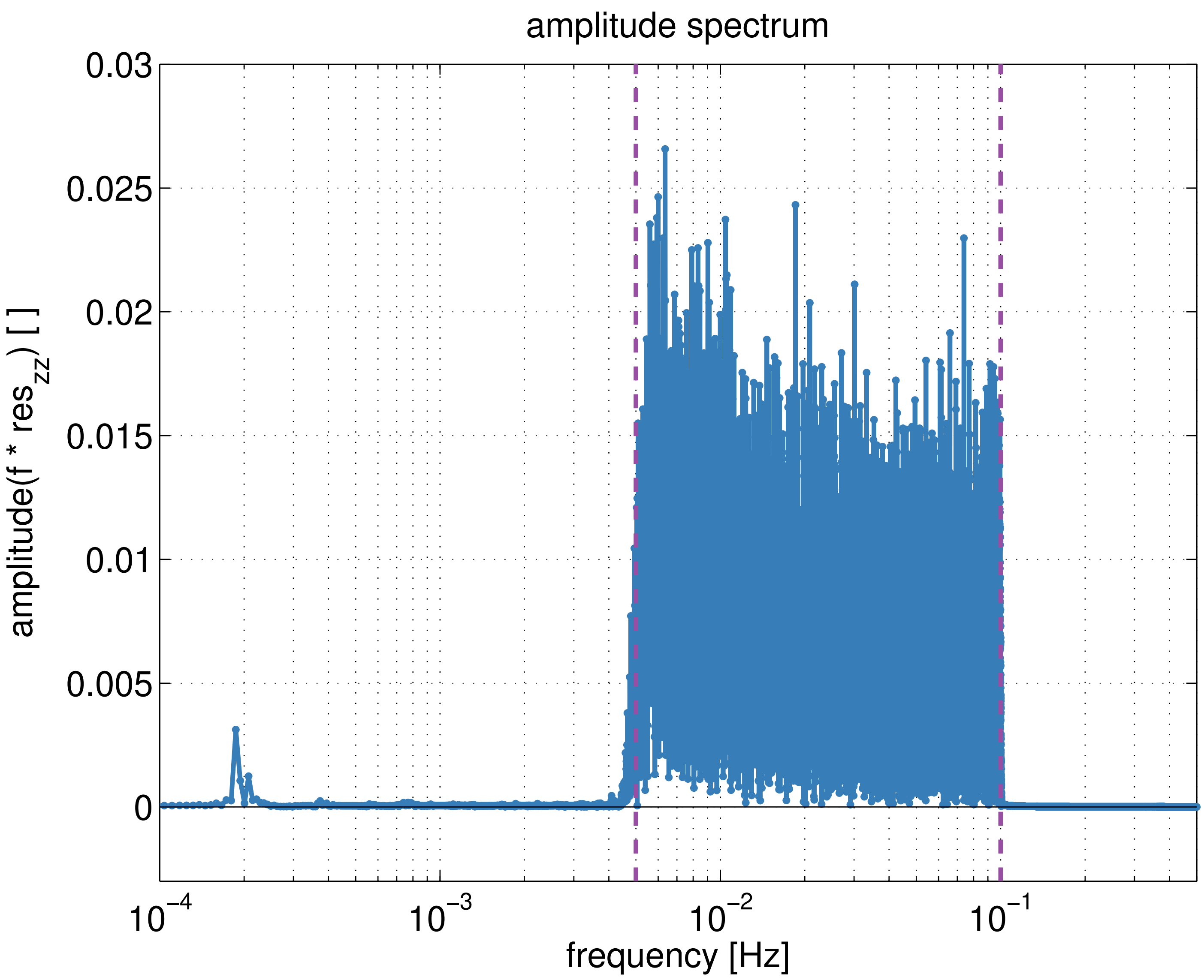






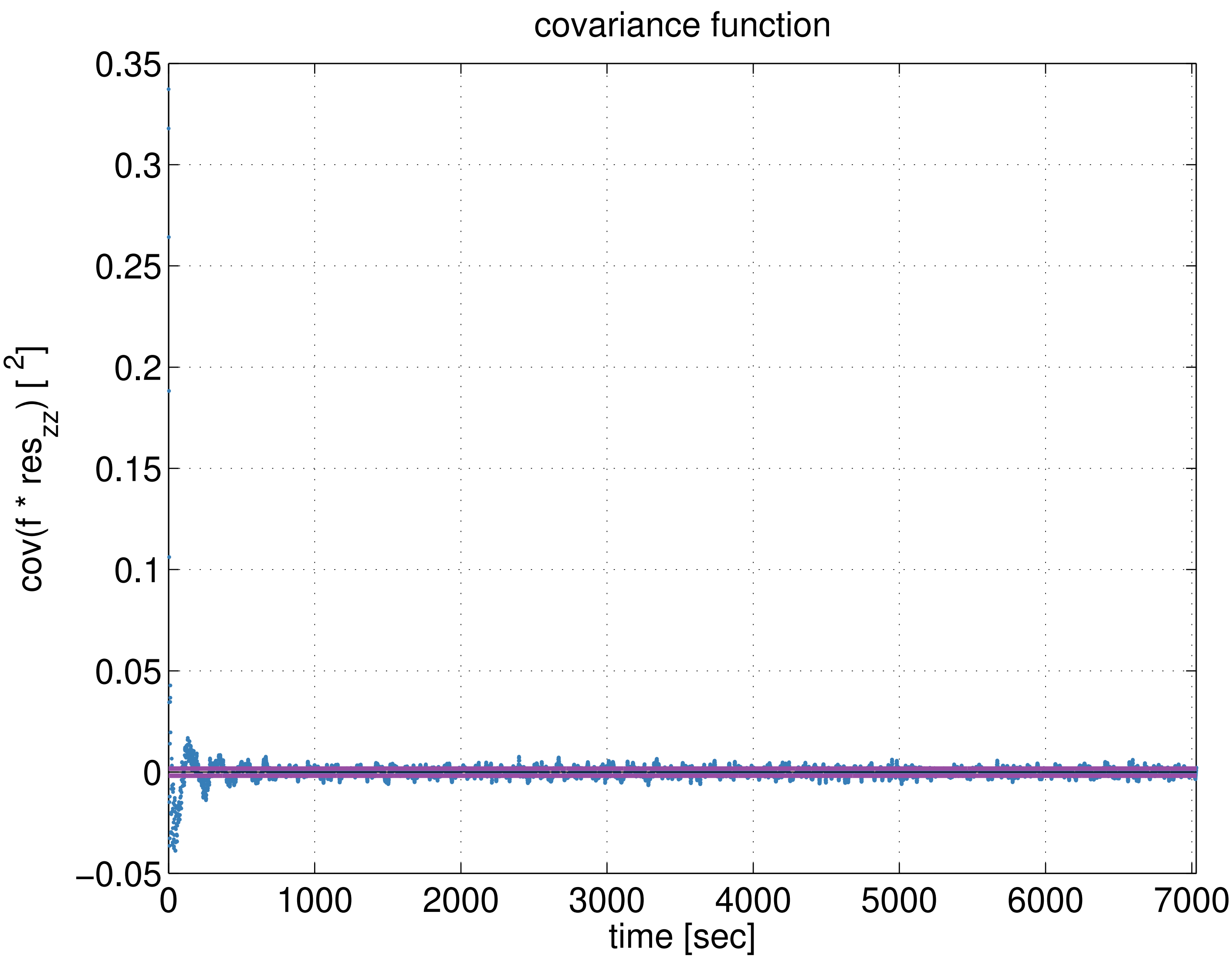
# GOCE - filter 9299





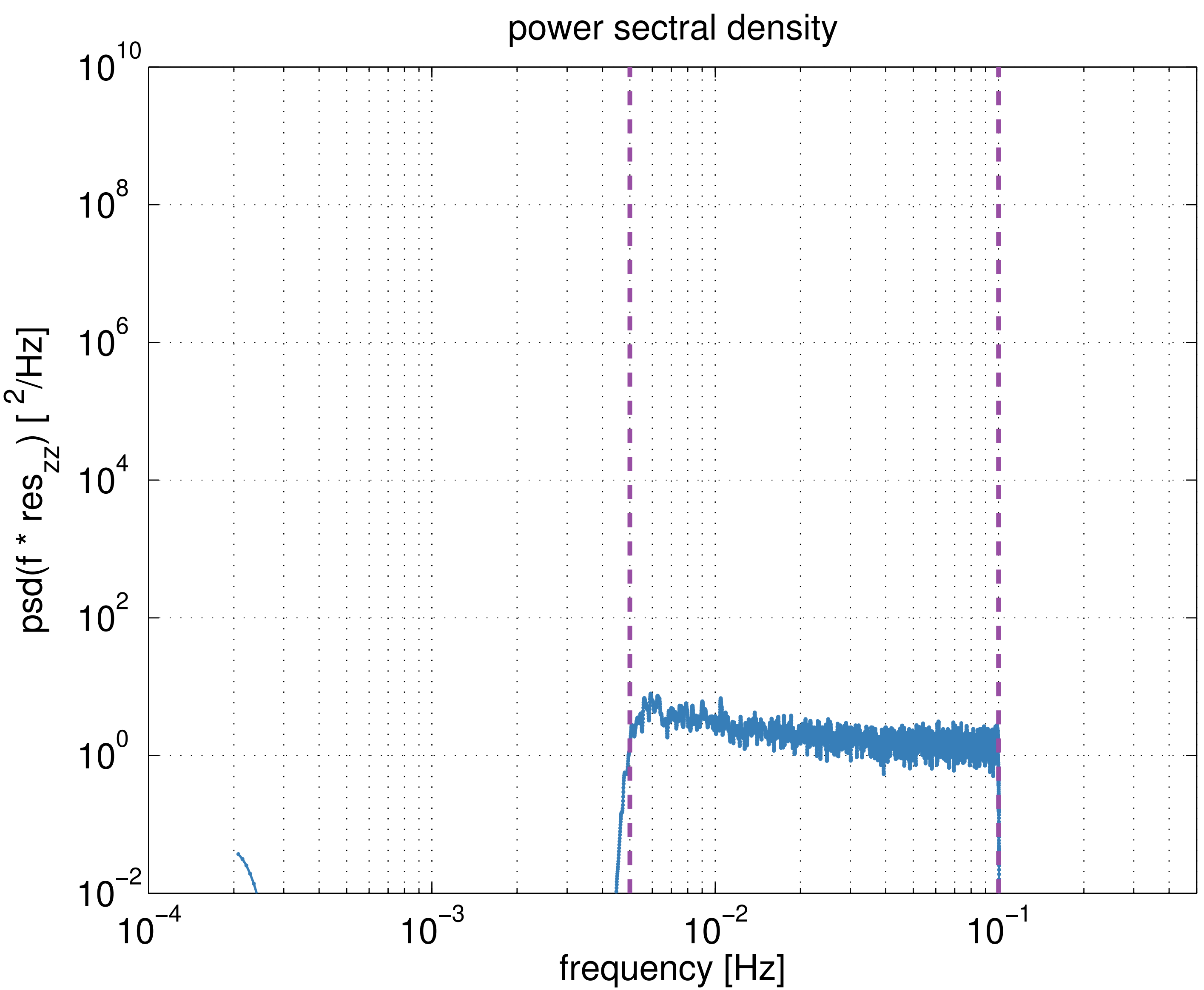
# GOCE - filter 9299





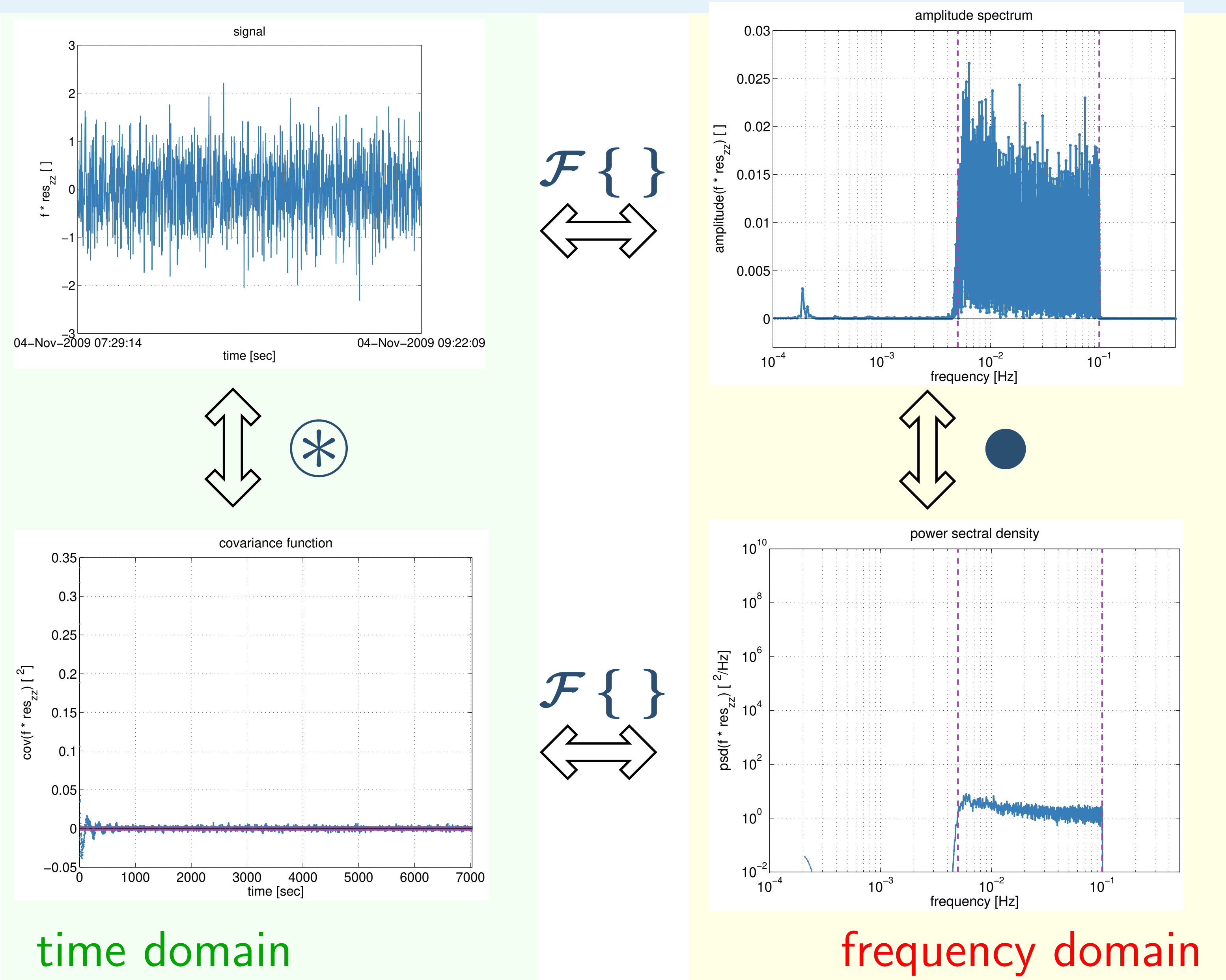
# GOCE - filter 9299





# GOCE - filter 9299

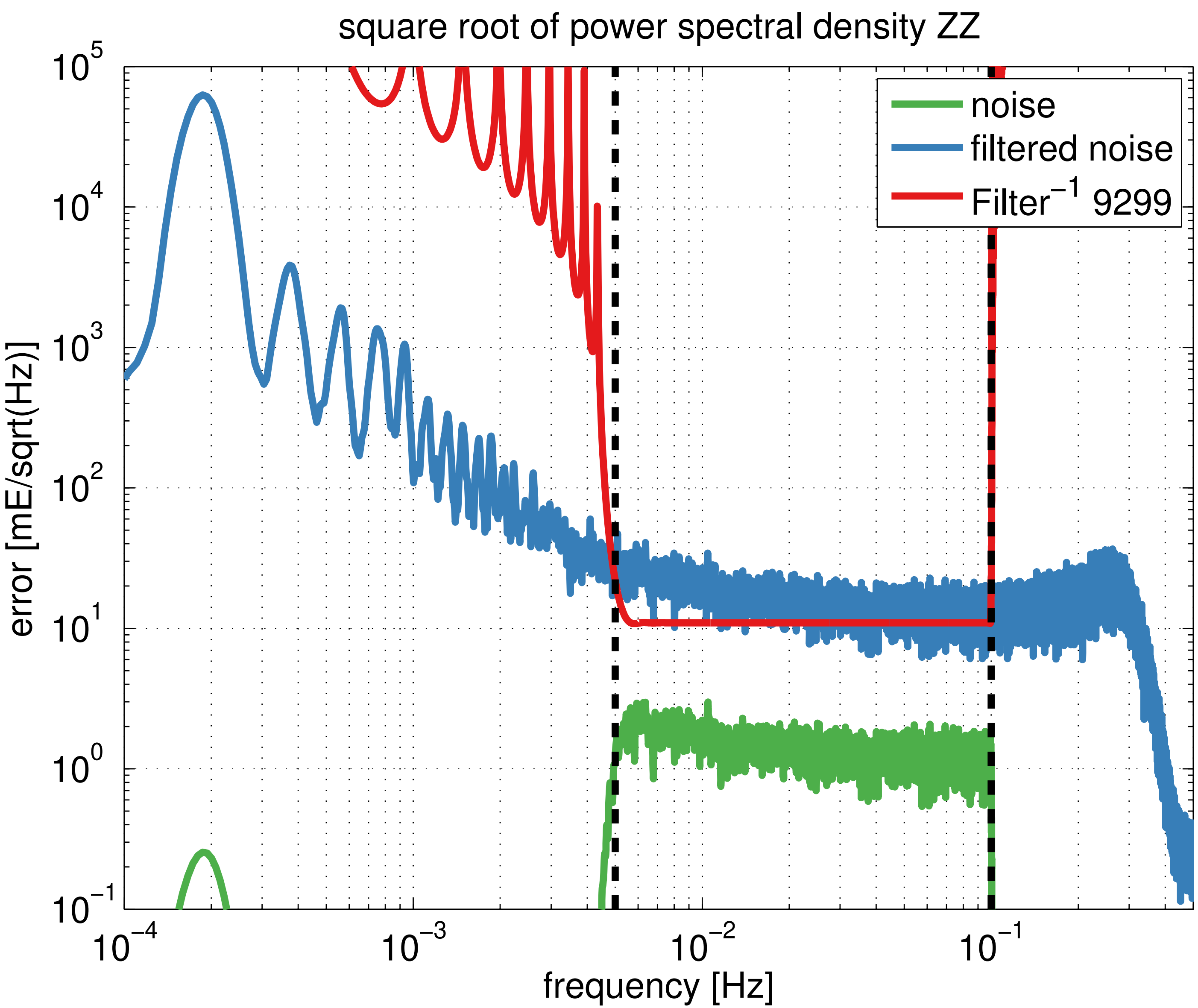




# GOCE - filter 9299





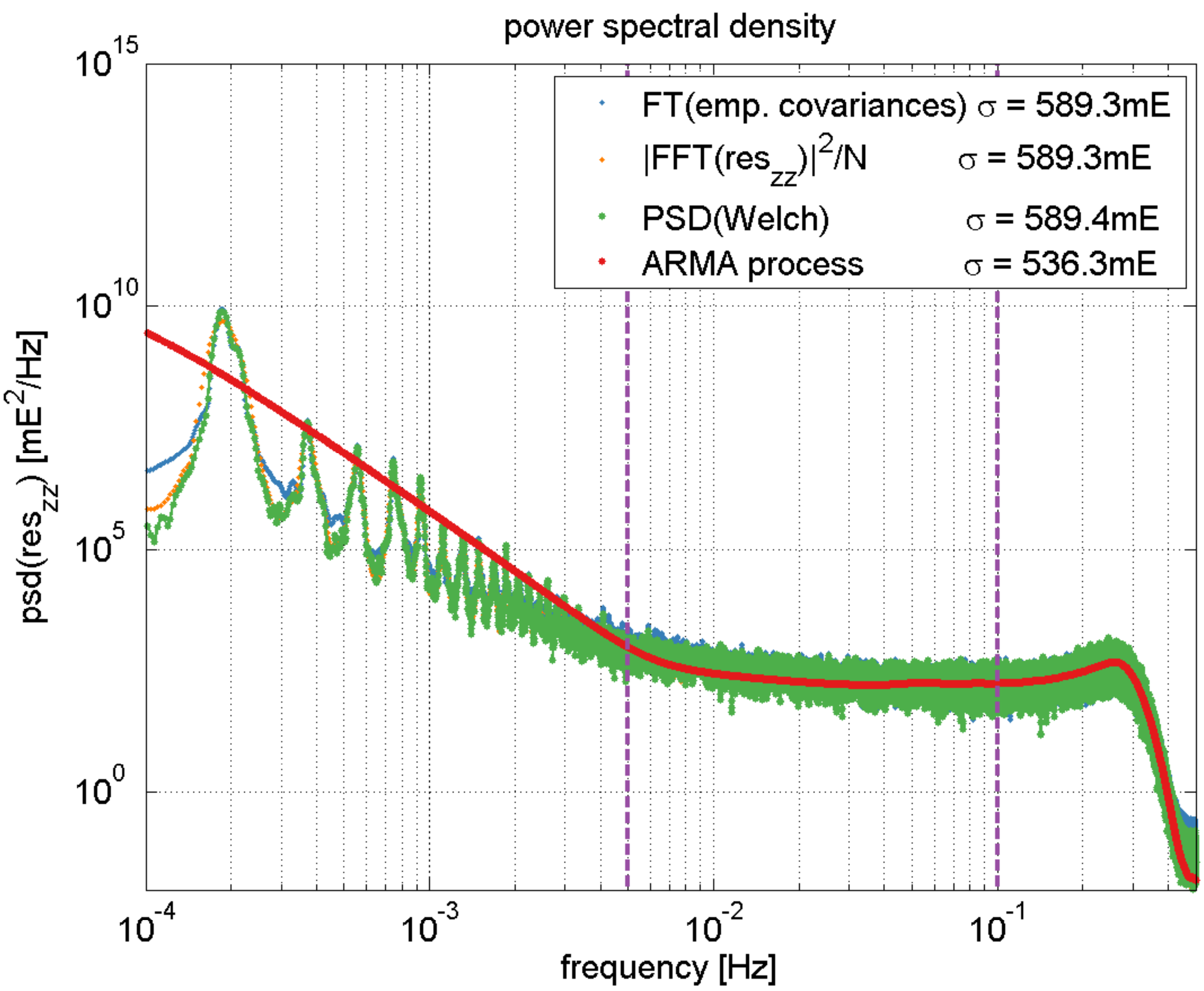


# GOCE - filter 9024

# GOCE - filter 9024

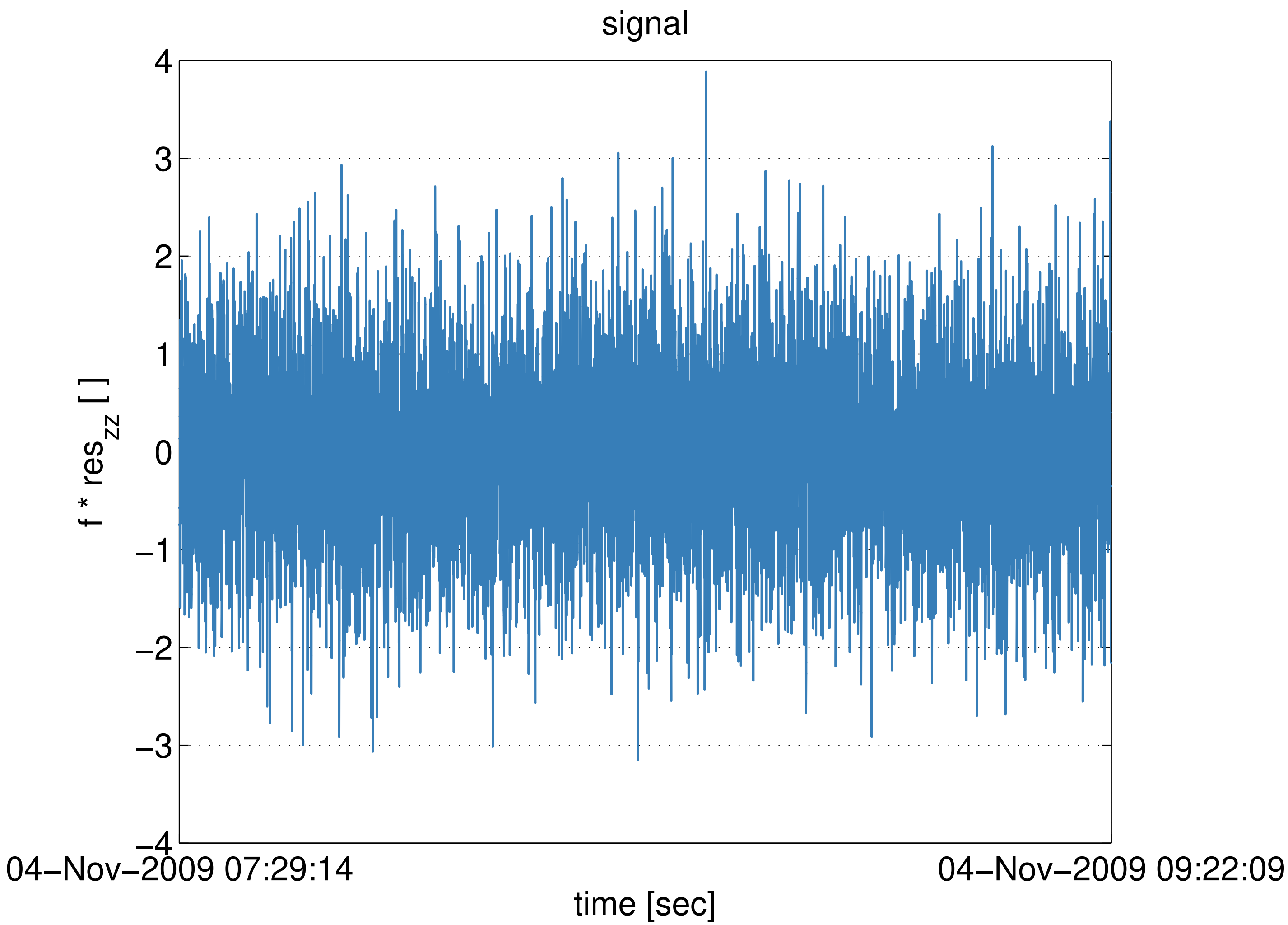






# GOCE - filter 9024

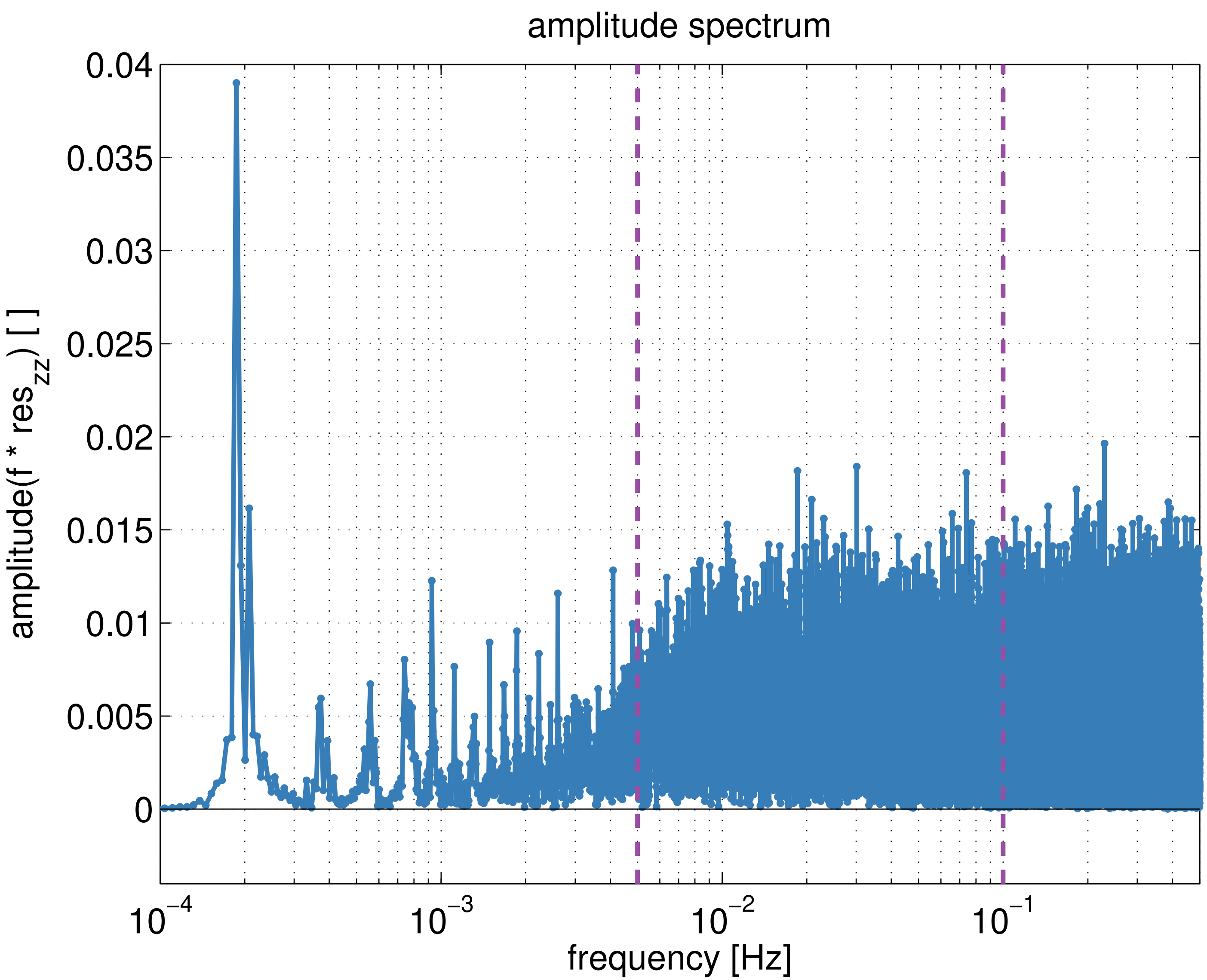


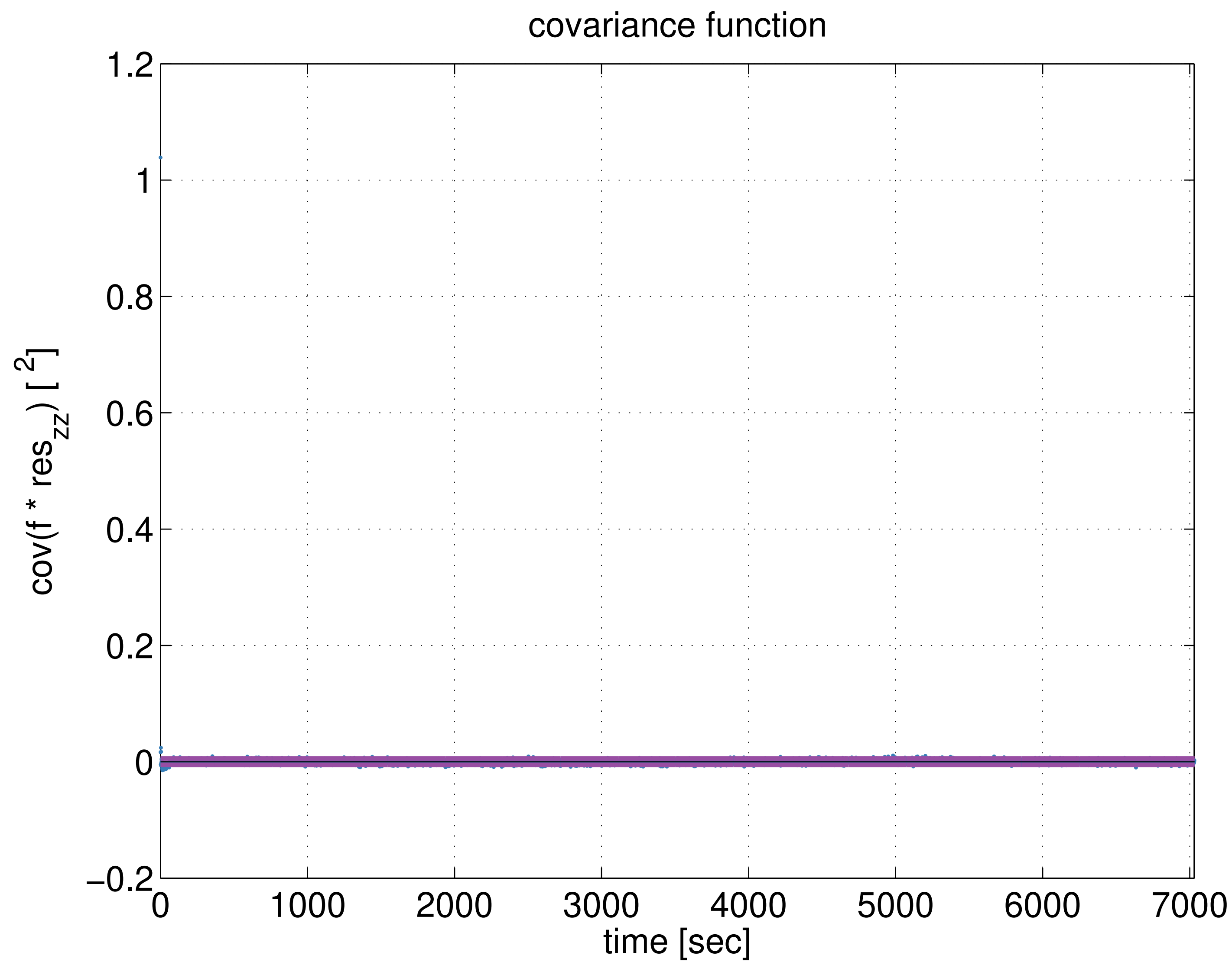


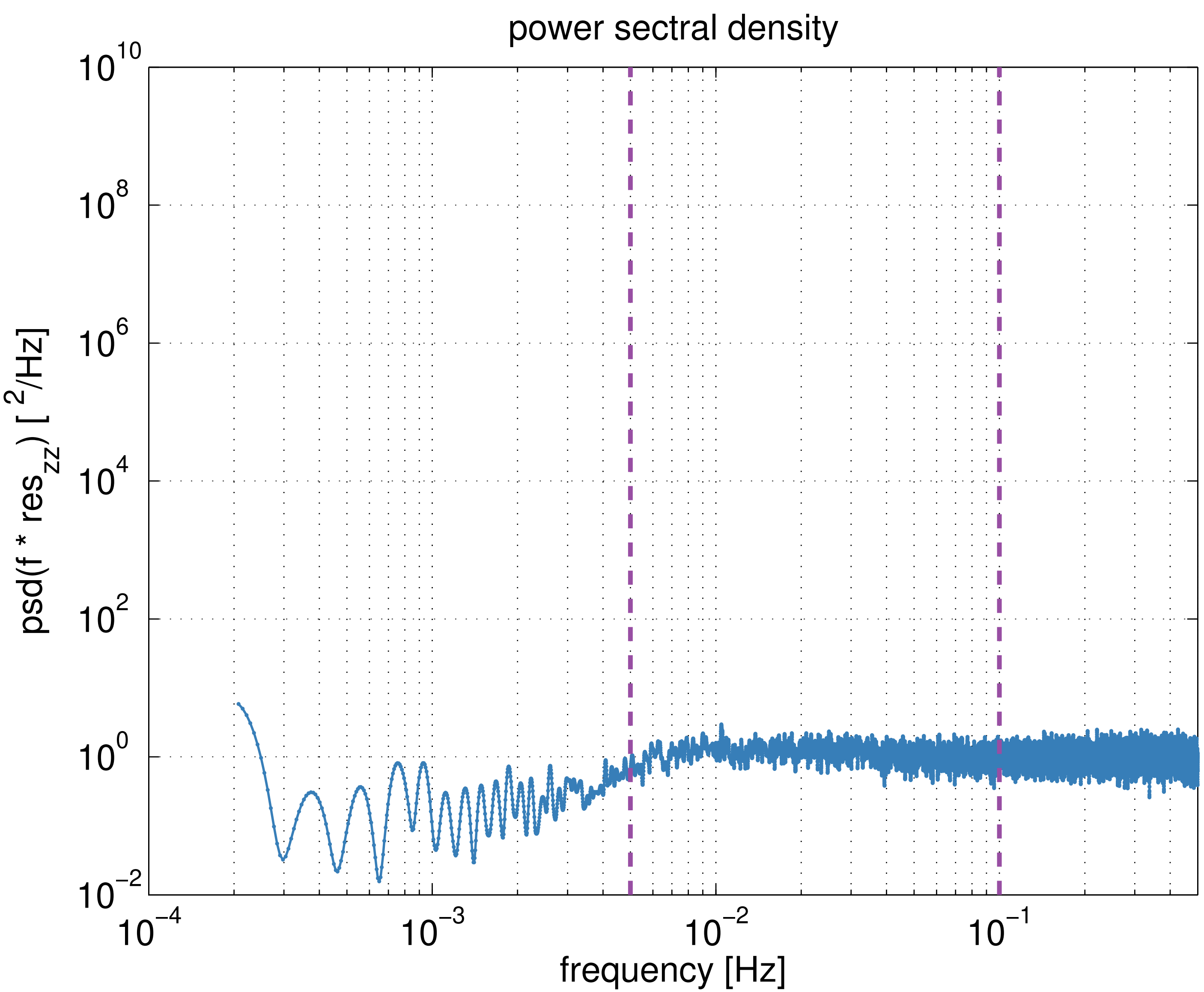
# GOCE - filter 9024





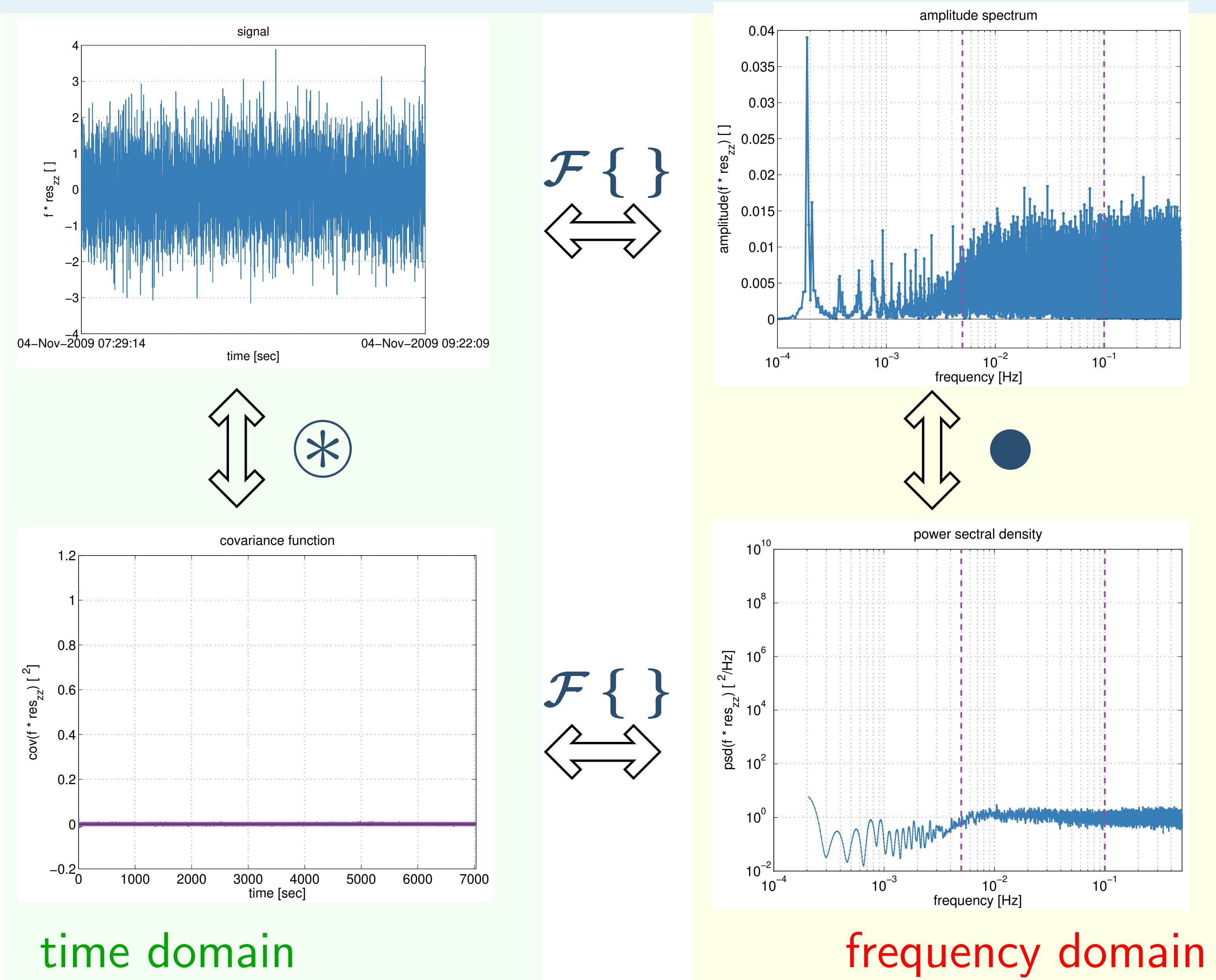



covariance function
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
cov(f * res_zz) [ 2]
0
1000
2000
3000
4000
5000
6000
7000
time [sec]


power sectral density
psd(f * res_zz) [ 2/Hz]
frequency [Hz]
10^10
10^8
10^6
10^4
10^2
10^0
10^-2
10^-4
10^-3
10^-2
10^-1

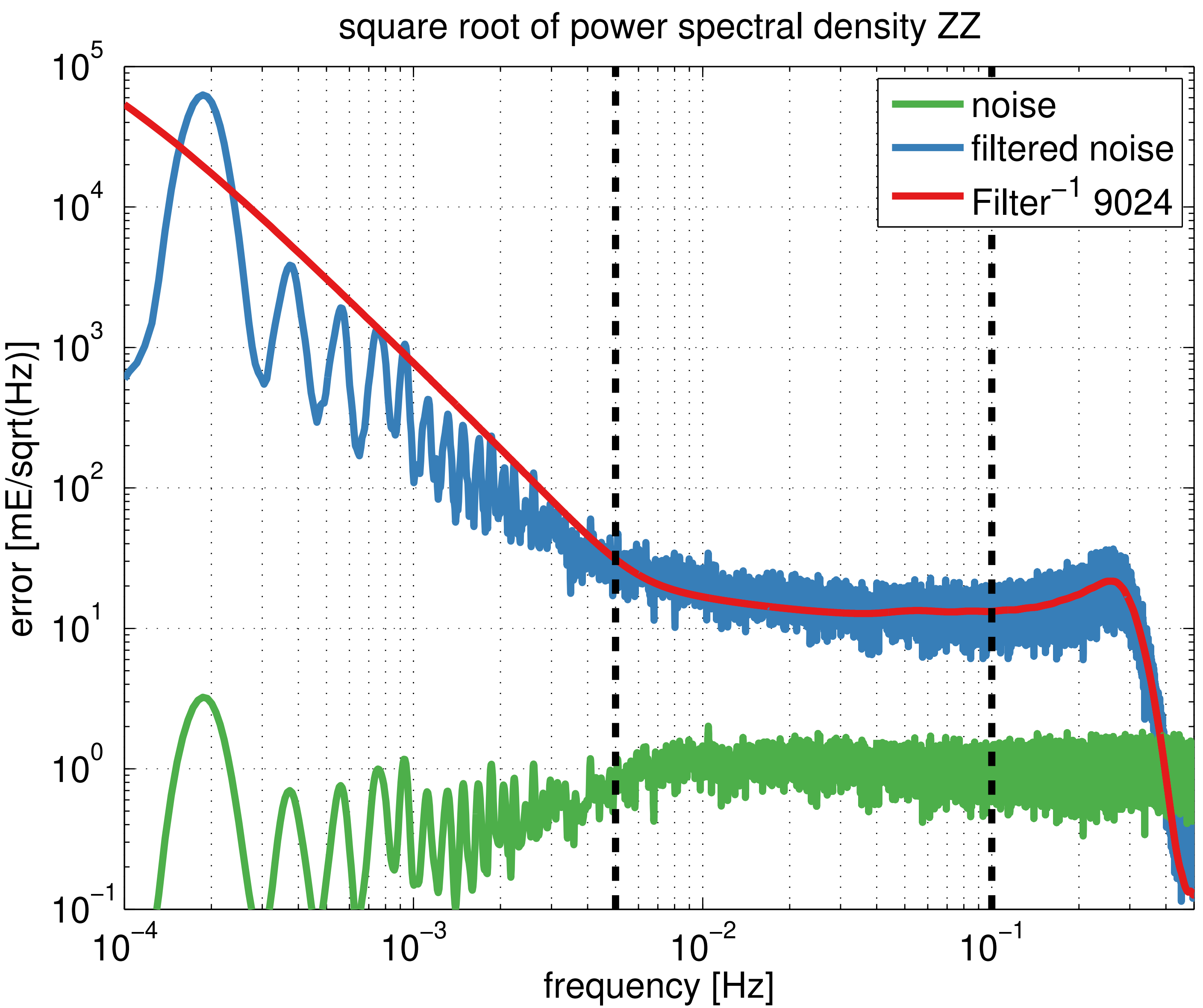
# GOCE - filter 9024





# GOCE - filter 9024

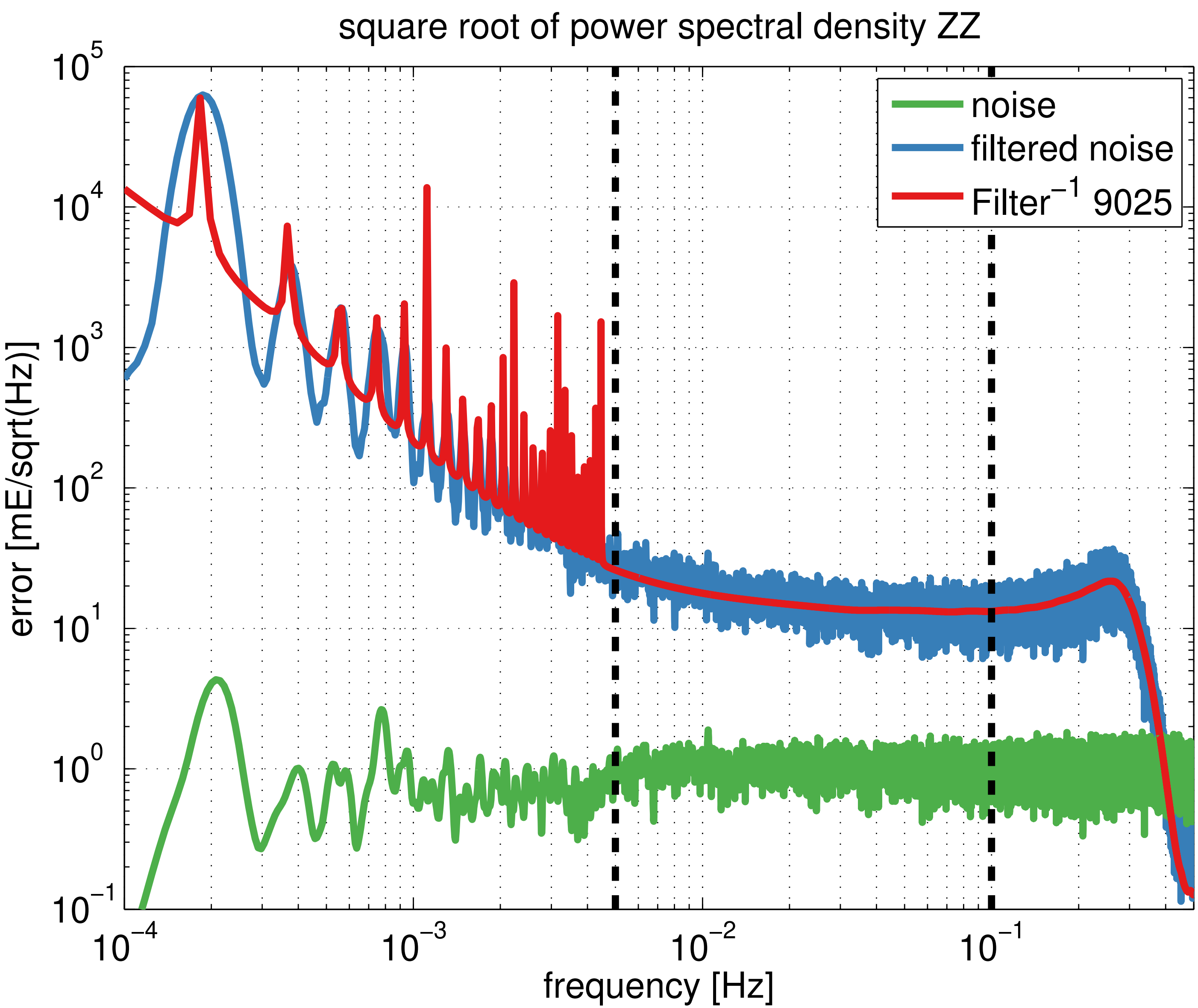




# GOCE - filter 9025







# Résumé

## Filter for . . .

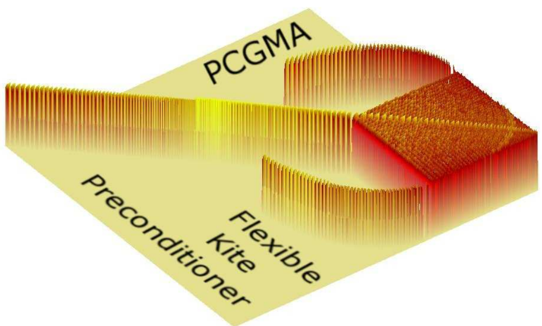
## Filter for . . .

- decorrelation - complex filter design

## Filter for . . .

- **decorrelation - complex filter design**

filter for tailored solution strategies




Preconditioned
Conjugate
Gradient
Multiple
Adjustment

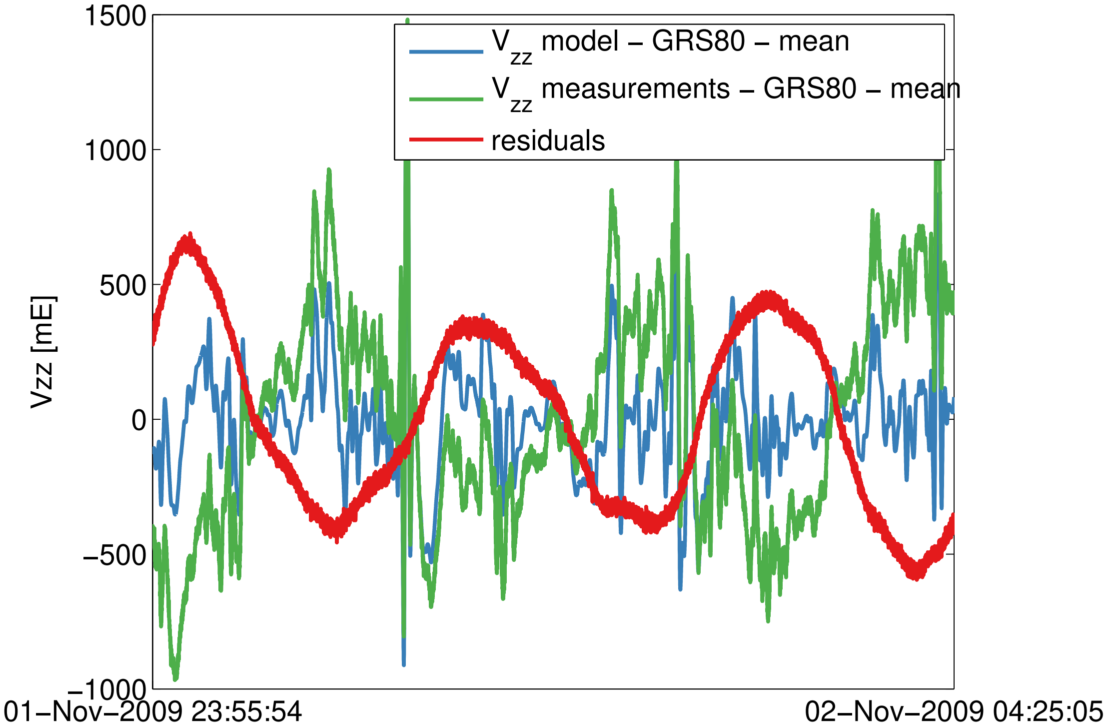
Juropa, Jülich - 2208x2x4

## Filter for . . .

- decorrelation - complex filter design
- data exploration - time/space conserving filters

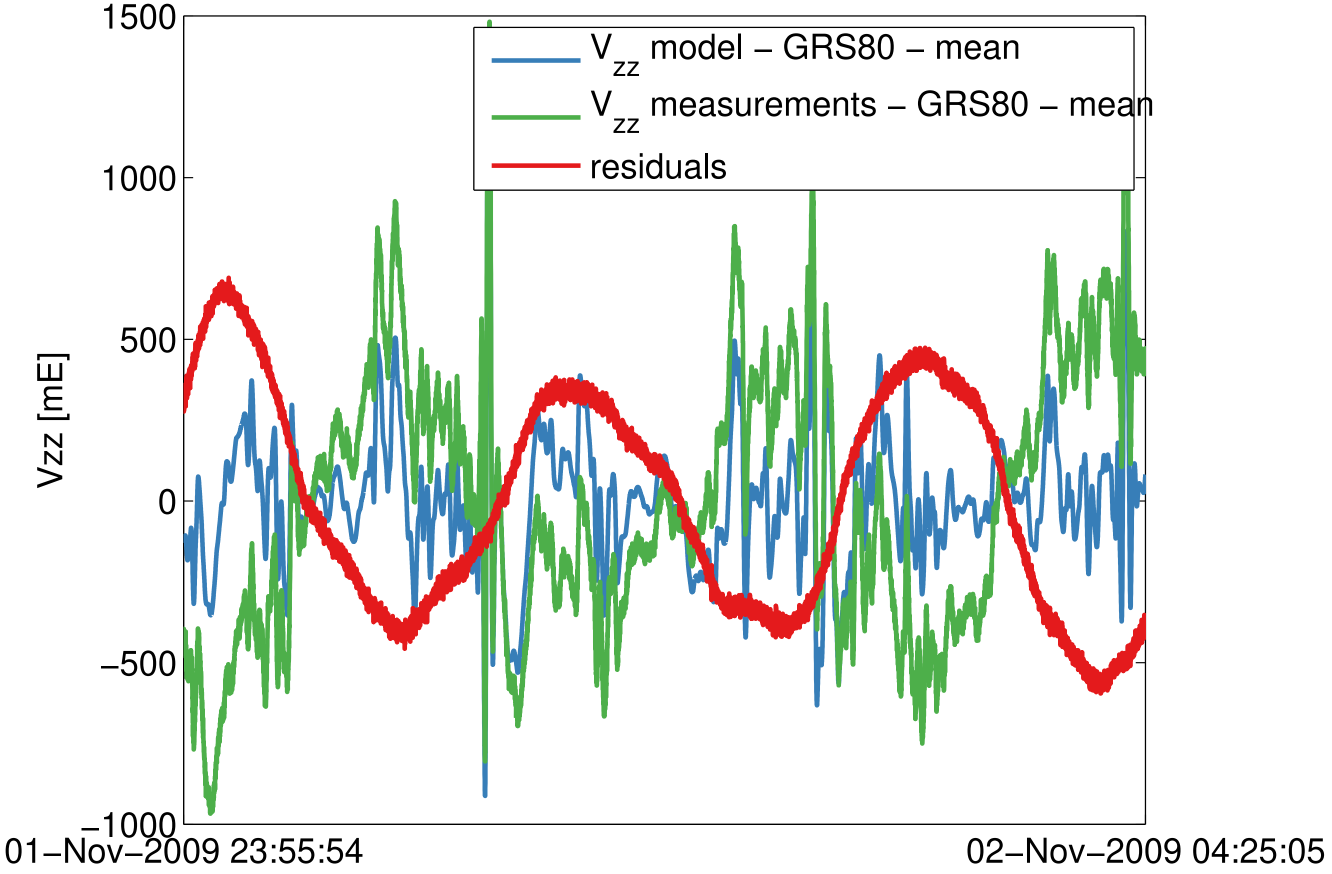
## Filter for . . .

- **decorrelation - complex filter design**
- **data exploration - time/space conserving filters**



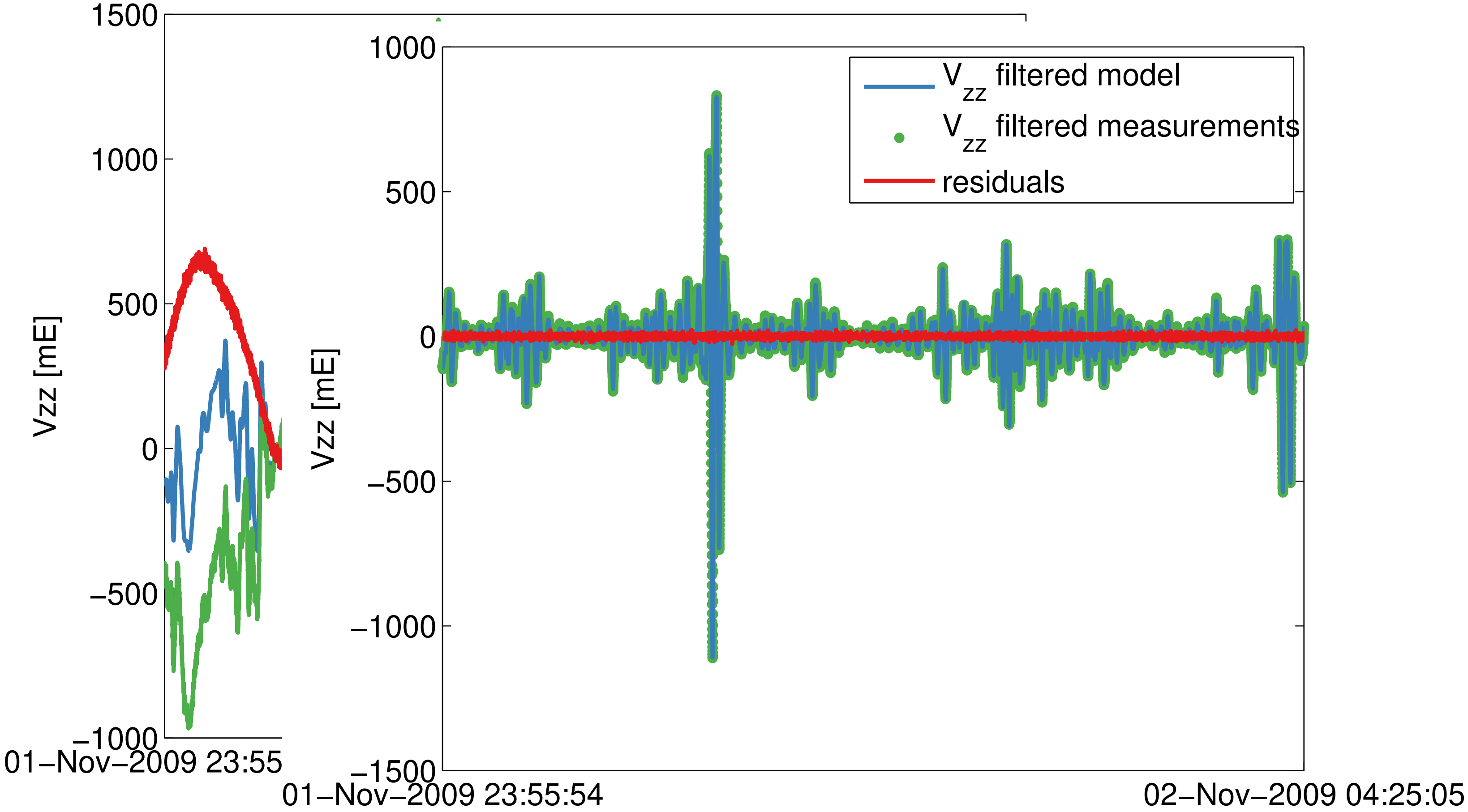
# Résumé





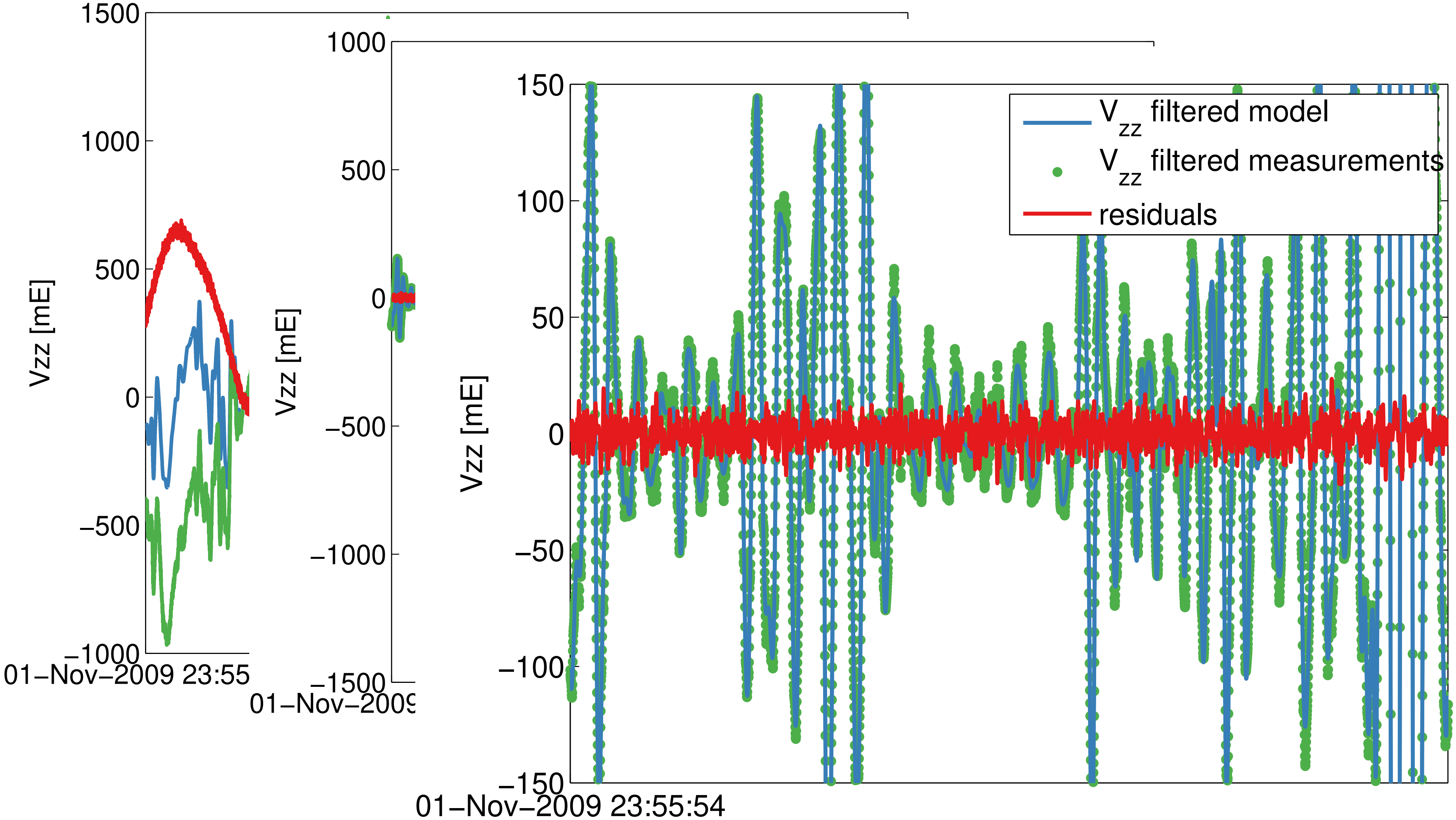
1500
1000
500
0
−500
−1000
Vzz [mE]
01−Nov−2009 23:55:54
02−Nov−2009 04:25:05
$V_{zz}$ model − GRS80 − mean
$V_{zz}$ measurements − GRS80 − mean
residuals

# Résumé

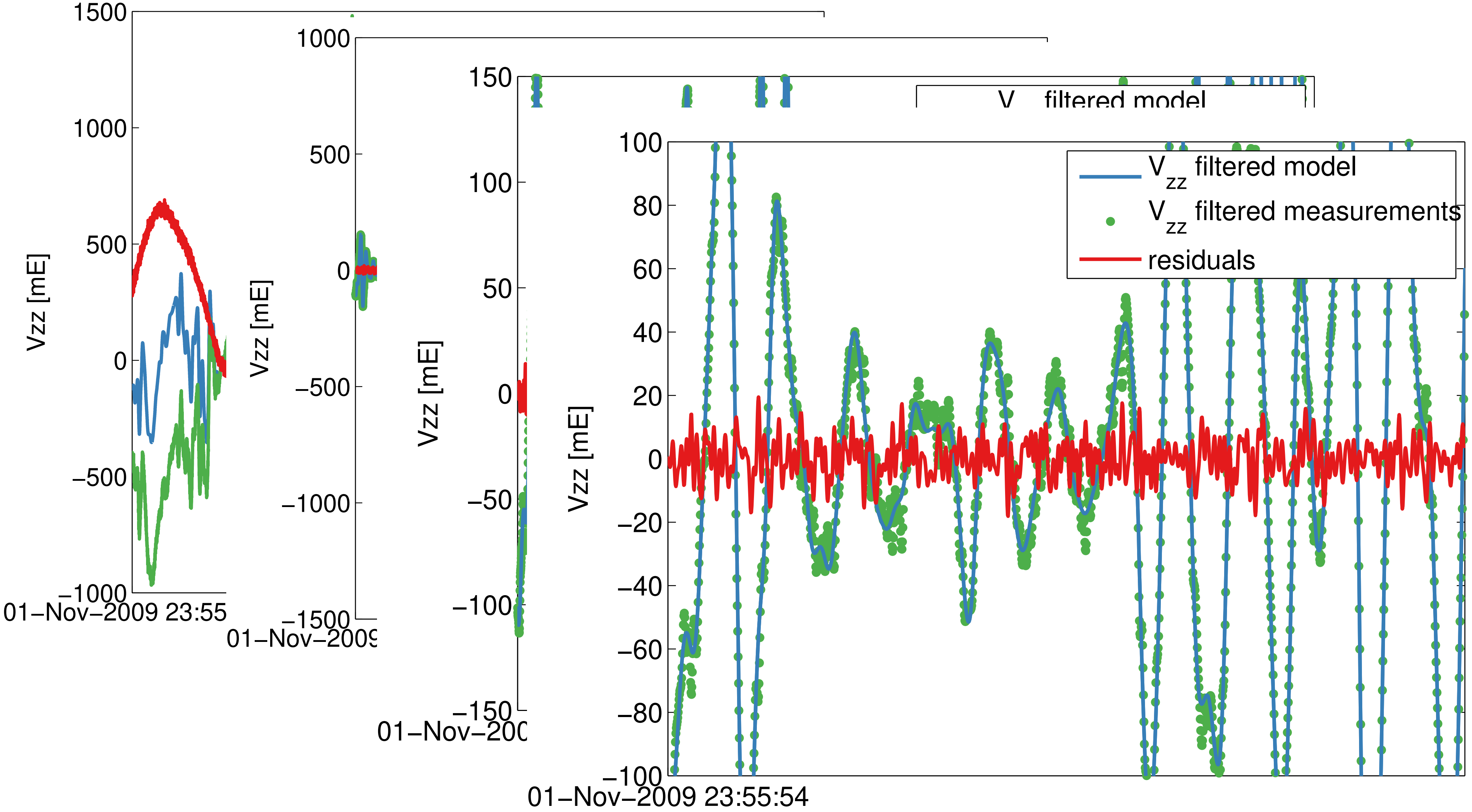

1500
1000
500
0
−500
−1000
Vzz [mE]
01−Nov−2009 23:55
1000
500
0
−500
−1000
−1500
Vzz [mE]
01−Nov−2009 23:55:54
02−Nov−2009 04:25:05
$V_{zz}$ filtered model
$V_{zz}$ filtered measurements
residuals

# Résumé

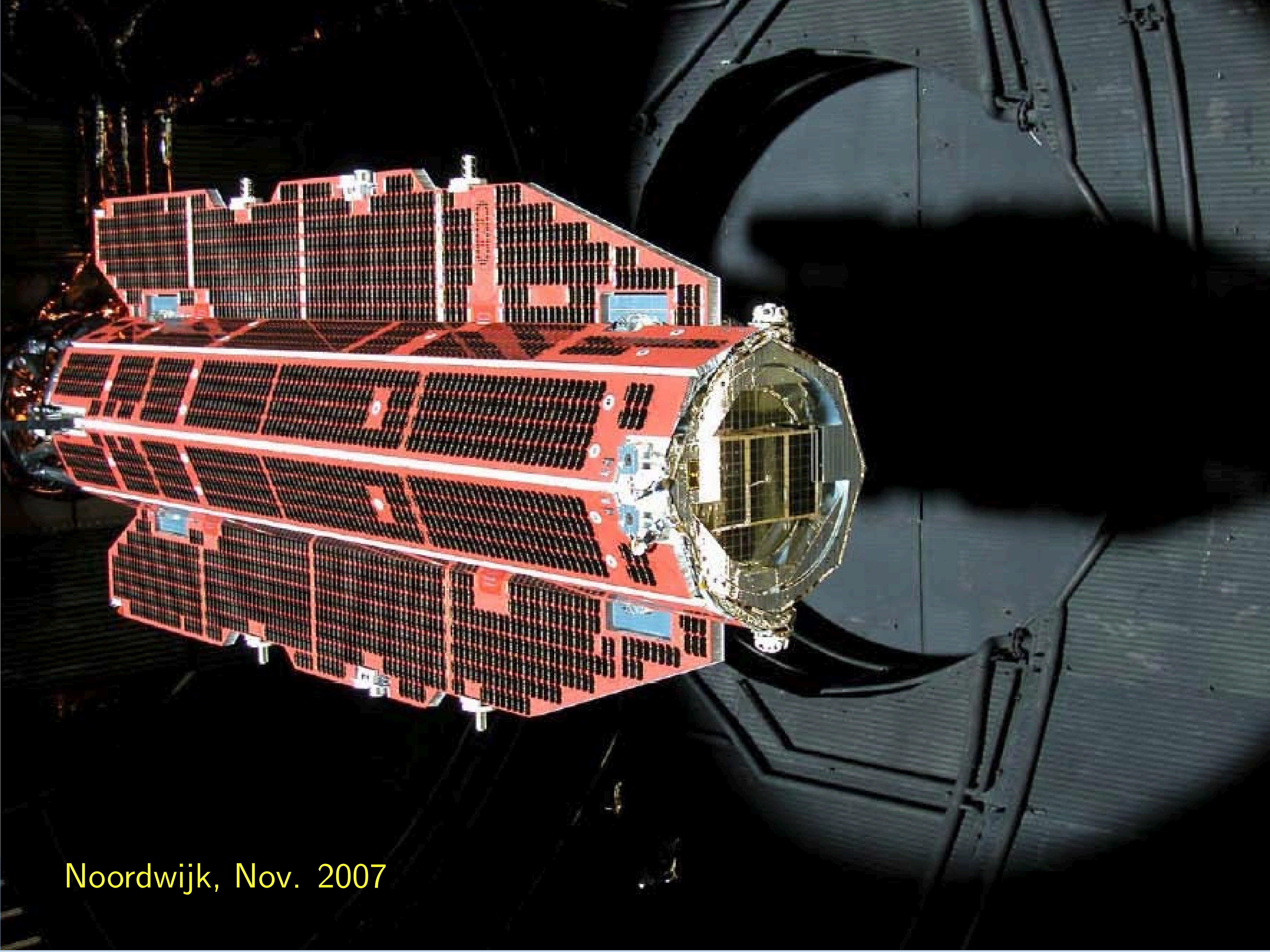



1500
1000
500
0
−500
−1000
Vzz [mE]
01−Nov−2009 23:55
1000
500
0
−500
−1000
−1500
Vzz [mE]
01−Nov−2009
150
100
50
0
−50
−100
−150
Vzz [mE]
01−Nov−2009 23:55:54
V zz filtered model
V zz filtered measurements
residuals

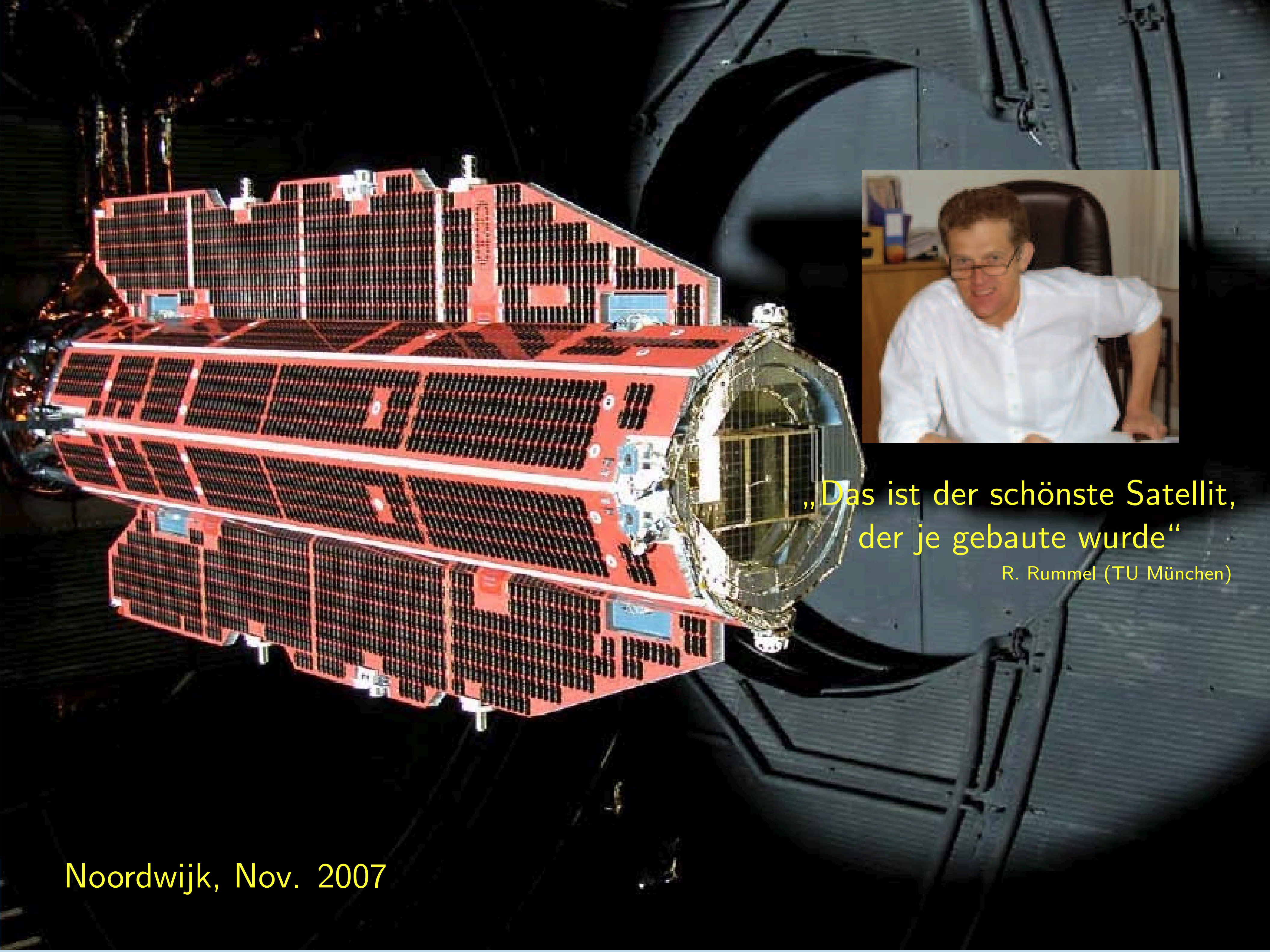
# Résumé



Vzz [mE]
01–Nov–2009 23:55
01–Nov–2009
01–Nov–200
01–Nov–2009 23:55:54
$V_{zz}$ filtered model
$V_{zz}$ filtered measurements
residuals



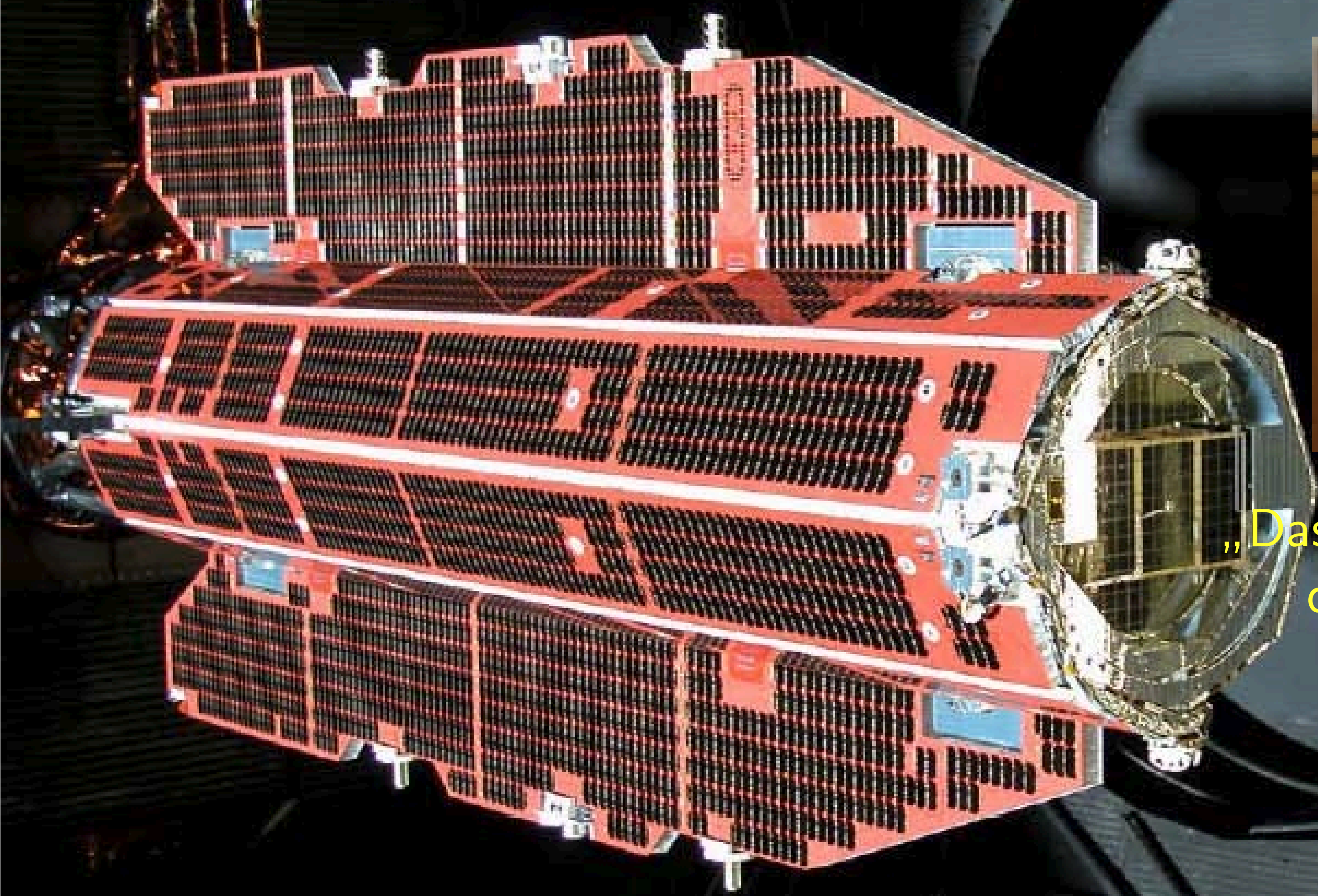
Noordwijk, Nov. 2007

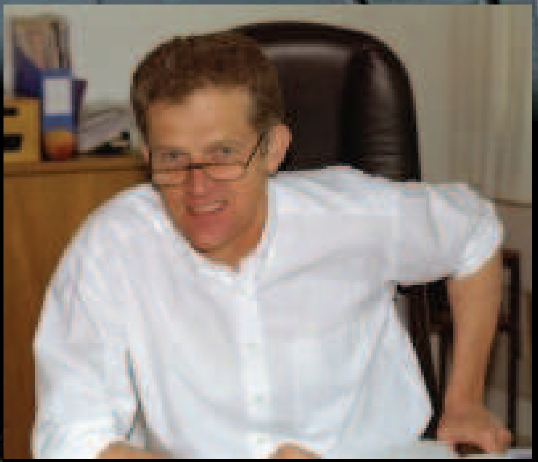


„Das ist der schönste Satellit,
der je gebaute wurde“

R. Rummel (TU München)

Noordwijk, Nov. 2007





„Das ist der schönste Satellit, der je gebaute wurde“

R. Rummel (TU München)

# Danke

www.esa.int/goce
earth.esa.int/goce

Noordwijk, Nov. 2007

# Literatur



**Digital Filters:**

[Hamming 1998] HAMMING, RICHARD W. (1998) *Digital Filters*. Third Edition. Dover Publications, Inc. Mineola, New York.

**Spectral Analysis:**

[Buttkus 2000] BUTTKUS, B. (2000) *Spectral Analysis and Filter Theory in Applied Geophysics*. Springer, Berlin.

[Papoulis 1977] PAPOULIS, A. (1977) *Signal Analysis*. International Student Edition. McGraw-Hill.

[Papoulis 1981] PAPOULIS, A. (1981) *Systems and Transforms with Applications in Optics*. R.E. Krieger Publishing Company, Malabar, Florida.

[Priestley 2001] PRIESTLEY, M.B. (2001) *Spectral Analysis and Time Series*. Academic Press.

[Schlittgen and Streitberg 1991] SCHLITTGEN, R. and B.H.J. STREITBERG (1991) *Zeitreihenanalyse*. Oldenbourg Verlag, München - Wien.

## Dissertations/Masterthesis/... IGG-TG:

[Alkhatib 2007] ALKHATIB, H. (2007) *On Monte Carlo methods with applications to the current satellite gravity missions*. Dissertation, Promotion an der Landwirtschaftlichen Fakultät der Universität Bonn, Schriftenreihe des Instituts für Geodäsie und Geoinformation der Rheinischen Friedrich-Wilhelms-Universät, Folge 7.

[Boxhammer 2006] BOXHAMMER, CH. (2006) *Effiziente numerische Verfahren zur sphärischen harmonischen Analyse von Satellitendaten*. Dissertation, Promotion an der Landwirtschaftlichen Fakultät der Universität Bonn, Mitteilungen aus den Geodätischen Instituten, Universität Bonn, Folge 94.

[Brockmann 2008] BROCKMANN, J.M. (2008) Effiziente Varianzkomponentenschätzung über iterative Techniken bei der GOCE-Daten Prozessierung. Diplomarbeit, Institut für Geodäsie und Geoinformation, Professur für Theoretische Geodäsie.

[Krasbutter 2009] KRASBUTTER, I. (2009) Dekorrelation und Daten-TÜV der GOCE-Residuen. Diplomarbeit, Institut für Geodäsie und Geoinformation, Professur für Theoretische Geodäsie.

[Ries 2003] Ries, J. (2003) Vergleichende Untersuchungen von Verfahren zur Schätzung von ARMA-Methoden zur Modellierung farbigen Rauschens. Diplomarbeit, Institut für Theoretische Geodäsie, Universität Bonn.

[Roese-Koerner 2009] Roese-Koerner, L. (2009) Quadratische Programmierung mit Ungleichungen als Restriktionen. Diplomarbeit, Institut für Geodäsie und Geoinformation, Professur für Theoretische Geodäsie.

[Schuh 1996] Schuh, W.-D. (1996) *Tailored numerical solution strategies for the global determination of the earth's gravity field.* Mitteilungen der Universität Graz, Graz. Folge 81.

[Siemes 2004] Siemes, Ch. (2004) Optimale Modellierung und Detektierung von Ausreißern in Zeitreihen. Diplomarbeit, Institut für Theoretische Geodäsie, Universität Bonn.

[Siemes 2008] Siemes, Ch. (2008) *Digital Filtering Algorithms for Decorrelation within Large Least Squares Problems.* Dissertation, Promotion an der Landwirtschaftlichen Fakultät der Universität Bonn.