# LEAST SQUARES ADJUSTMENT
# A MODERN APPROACH

by
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**Part A:** Algebraic and Geometric Approach
Toward Least Squares Adjustment

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# Preface

For his lectures at the Tongji University in Shanghai and at other institutions in China in November - December 1981, Peter Meissl prepared a set of lecture notes on contemporary least-squares adjustment and applications. Subsequently he worked on correcting and expanding them, but this was interrupted by his tragic death on May 22, 1982. (For Peter Meissl's life and work, the reader is referred to his biography by Franz Allmer, Mitteilungen der geodätischen Institute der Technischen Universität Graz, Folge 44, 1983.)

In view of the unique importance of this work, the Institute of Theoretical Geodesy decided to edit the manuscript posthumously and to publish the book in the series of the Geodetic Institutes of the Technical University, Graz, although Peter Meissl himself would certainly have included additional topics such as inner adjustment theory, expanded others such as the theory of large networks, and polished the manuscript much more before being satisfied with its publication.

The finishing of the book is due to Peter Meissl's closest associates: Dr. Norbert Bartelme, Dr. Helmut Fuchs, Dr. Bernhard Hofmann-Wellenhof, Dipl.-Ing. Wolf-Dieter Schuh and Dipl.-Ing. Manfred Wieser. In addition to being responsible for the careful editing of the manuscript, they also prepared the printing text using the word processing facilities of the computer WANG 2200 MVP.

A glance at the table of contents shows that this book is a thoroughly modern text on least-squares adjustment. In the contemporary spirit, the usual linear algebra is treated in the context of general linear spaces, which makes possible an easy transition to Hilbert space important for advanced topics. Also modern is the division into an algebraic and geometric approach (without statistics) and a stochastic approach, including statistical tests. Applications to Doppler observations, large networks, geodetic data bases, and splines essentially increase the practical usefulness. Although the book develops adjustment theory in a systematic and self-contained way, it will be best appreciated by readers who already have some elementary previous knowledge of adjustment computations.

The book needs no recommendation. Both students and research workers will find it indispensable. It is a fitting memorial of a great scientist.

Helmut Moritz

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# A. THE ALGEBRAIC AND GEOMETRIC APPROACH TOWARD LEAST SQUARES ADJUSTMENT

## 1. Vector spaces.

### 1.1. Definition.

A real vector space (also called real linear space) is a set of elements, called vectors, having the following properties. If $a_1,\ldots,a_m$ are vectors of the vector space V, and if $\lambda_1,\ldots,\lambda_m$ are real numbers, then the linear combination

$$\lambda_1 a_1 + \ldots + \lambda_m a_m$$

must be defined and must be an element of V.

Remark. The above definition is logically not complete. A set of familiar computational rules must be postulated: $\lambda(a+b) = \lambda a + \lambda b$, $(\lambda+\mu)a = \lambda a + \mu a$, $\lambda(\mu a) = (\lambda\mu)a$, $1a = a$. By the way, the expression $\lambda a$ may equally well be written as $a\lambda$.

It is seen that in a vector space essentially two mathematical operations are available, multiplication of a vector by a scalar, and addition of two vectors. The neutral element of scalar multiplication is the real number 1. The neutral element of addition is the zero vector. It is obtained either as $0a$ or as $a-a$.

Remark on notation. In the first sections we shall consistently use upper case Latin letters for vector spaces, lower case Latin letters for vectors, and lower

case Greek letters for scalars. In later chapters the rather sparse notational resources of the western world must be allocated differently.

## 1.2. Examples of vector spaces.

1.2.1. R, the real line is a vector space.

1.2.2. $R^n$, the set of n-tuples

$$a = (\alpha_1, \ldots, \alpha_n)$$

forms a vector space. The real numbers $\alpha_i$ are called components. Scalar multiplication and addition are defined component-wise in an obvious and familiar way.

1.2.3. The set of all linear forms

$$a = \alpha_1 \xi_1 + \ldots + \alpha_n \xi_n$$

in n variables $\xi_1, \ldots, \xi_n$, is a vector space.

1.2.4. The set of all polynomials

$$a = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 \ldots + \alpha_n \xi^n$$

in one variable $\xi$ forms a vector space. The $\alpha_i$ are called coefficients. Note

that multiplication of two polynomials does not correspond to a characteristic structural property of a vector space. It is an additional feature of spaces of polynomials which is exploited in polynomial algebra.

1.2.5. The set of continuous functions $f(\xi)$ defined on an interval $\alpha \leq \xi \leq \beta$ is a vector space. The scalar multiple of a continuous function is continuous; so is the sum of two continuous functions.

1.2.6. The set of all solutions to a linear homogeneous system

$$\begin{aligned}
\alpha_{11}\xi_1 + \ldots + \alpha_{1m}\xi_m &= 0 \\
\alpha_{21}\xi_1 + \ldots + \alpha_{2m}\xi_m &= 0 \\
&\ldots\ldots\ldots\ldots\ldots \\
\alpha_{n1}\xi_1 + \ldots + \alpha_{nm}\xi_m &= 0
\end{aligned}$$

1.2.7. A subset U of a vector space V may be a vector space by itself. Such a subset U is called a (vector-) subspace of V. An example is the set of all polynomials of degree $\leq n$. This set is a subspace of the space of all polynomials introduced in 1.2.4. In turn, the space of all polynomials may be seen as a subspace of the space of continuous functions.

1.2.8. If V is a vector space, and if $a_1, \ldots, a_m$ are vectors in V, then the set U of all linear combinations

$$\lambda_1 a_1 + \ldots + \lambda_m a_m$$

is a vector space. It is called the linear span of $a_1,\ldots,a_m$. It is a subspace of V. In symbols

$$U = \text{span}(a_1,\ldots,a_m)$$

It will be important to investigate conditions under which U=V.

1.2.9 Translations in the plane. Consider the two-dimensional plane as the set of its points. The points are not vectors. There is no meaningful way to define e.g. the sum of two points. Thus, the plane considered as the set of its points is not a vector space. Consider now a common translation of all points in the plane. All points move the same distance, and in the same direction along parallel lines. Such a translation is represented by an arrow. An arrow has a direction and a length. If we want to know the image of a point P under the translation we place the tail of the arrow at P and its tip will show the new position P'. The translations form a vector space. A translation can be multiplied by a scalar in an obvious way; two translations can be added according to the familiar parallelogram-rule. Thus translations can be linearly combined. See figure 1.1.

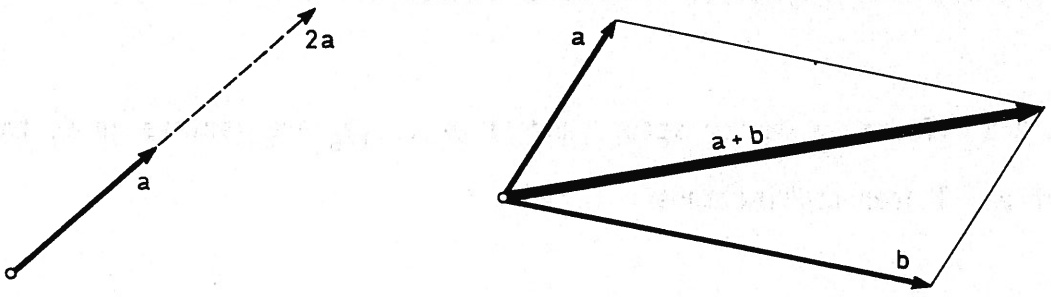



Fig. 1.1 Vectors as arrows

If one chooses an arbitrary point O as a reference point, then any point P in the plane is uniquely identified by the vector translating O into P. This vector is called position vector. In this way the plane is mapped onto the vector space (of translations). In sloppy, but convenient language, one then identifies the plane with this vector space. One should, however, bear in mind that this identification relies on the arbitrary choice of the reference point O.

1.3. Linear dependence and independence.

The vectors $a_1,\ldots,a_n$, are called linearly dependent if there exist scalars $\lambda_1,\ldots,\lambda_n$, not all equal to zero, such that

$$\lambda_1 a_1 + \ldots + \lambda_n a_n = 0$$

Linear dependence of vectors means that the zero vector can be obtained by a nontrivial linear combination.

Vectors $a_1,\ldots,a_n$ which are not linearly dependent are called linearly independent. The zero vector can only be obtained by the trivial linear combination. Such a linear combination has all $\lambda$'s equal to zero.

It is seen that the vectors $a_1,\ldots,a_n$ are linearly independent if

$$\lambda_1 a_1 + \ldots + \lambda_n a_n = 0$$

implies

$$\lambda_1 = \ldots = \lambda_n = 0$$

If a vector b is a linear combination of vectors $a_1,\ldots,a_m$:

$$b = \lambda_1 a_1 + \ldots + \lambda_m a_m$$

then b is called linearly dependent of $a_1,\ldots,a_m$. The vector b is then a member of span$(a_1,\ldots,a_m)$. The m+1 vectors $a_1,\ldots,a_m,b$ are necessarily linearly dependent.

1.4. Bases.

1.4.1. Definition. A set of linearly independent vectors $e_1,\ldots,e_n$, is called a basis of V if any vector x in V can be expressed as a linear combination

$$x = \xi_1 e_1 + \ldots + \xi_n e_n$$

The numbers $\xi_1,\ldots,\xi_n$ are called coordinates of the vector x with respect to the basis $e_1,\ldots,e_n$. It follows that V is spanned by the linearly independent vectors $e_1,..,e_n$:

$$V = \text{span}(e_1,\ldots,e_n)$$

The coordinates of a vector with respect to a basis are unique. For suppose that

$$x = \xi_1 e_1 + \ldots + \xi_n e_n$$
$$x = \xi'_1 e_1 + \ldots + \xi'_n e_n$$

By subtracting the two equations a linear combination yielding the zero vector is obtained. The coefficients of this linear combination are $(\xi_i - \xi'_i)$, $i=1,\ldots,n$. From the linear independence of the basis vectors one infers that $\xi_i = \xi'_i$, $i=1,\ldots,n$.

1.4.2. Finite dimensional vector spaces. A vector space having a finite basis is called finite dimensional. The choice of a basis is not unique, however the number of vectors in a basis is unique. It is called the dimension of V. The proof of the uniqueness of the dimension is not entirely trivial. If two bases $e_1,\ldots,e_n$ and $e'_1,\ldots,e'_m$ are given in V, and if $n \leq m$ is assumed, one can successively exchange unprimed vectors against primed vectors until a basis of n primed vectors is obtained. The details of the proof are omitted.

1.4.3. Examples of bases. (Confer section 1.2 on examples of vector spaces.)

1.4.3.1. Any nonzero number of R forms a basis of R. If the vector 1 is chosen as basis, any vector has a coordinate equal to itself. Hence 1 is called the natural basis of R.

1.4.3.2. The vector space $R^n$ of n-tuples introduced in 1.2.2 also possesses a

natural basis. It is given by

$$e_1 = (1,0,\ldots,0)$$
$$e_2 = (0,1,\ldots,0)$$
$$\ldots\ldots\ldots\ldots$$
$$e_n = (0,0,\ldots,1)$$

The coordinates $\alpha_i$ of an n-tuple a are then identical to the components of a.

1.4.3.3. The polynomials $1,x,x^2,\ldots,x^n$ form a natural basis of the space of polynomials of degree $\leq n$. The coordinates of a polynomial are then equal to its coefficients.

1.4.3.4. The space of all polynomials and the space of continuous functions over $\alpha \leq \xi \leq \beta$ do not have a finite bases. These spaces are infinite dimensional.

1.4.3.5. Two arrows having neither the same nor opposite directions represent a basis for the arrows (translations) in the plane.

1.4.4. Isomorphism between all vector spaces of dimension n. If $V_n$ is a general n-dimensional vector space, and if a basis $e_1,\ldots,e_n$ is chosen, a correspondence between $V_n$ and $R^n$ is established. Remember that coordinates are unique. Hence any vector in $V_n$ is uniquely mapped onto an n-tuple in $R^n$. The converse is trivially also true. The basis vectors of $V_n$ are mapped onto the natural basis vectors of $R^n$. The mapping between $V_n$ and $R^n$ preserves the linear structure:

Linear combinations are mapped onto linear combinations with identical scalar coefficients. If $x,y \in V$ have coordinates $\xi_i$, $\eta_i$, $i=1,\ldots,n$, then $\lambda x+\mu y$ has coordinates $\lambda\xi_i+\mu\eta_i$. In view of the preservation of the linear structure, the mapping is called an isomorphism.

It is seen that all n-dimensional vector spaces are isomorphic to $R^n$. It suffices to study the structure of $R^n$ in order to learn everything about finite dimensional vector spaces.

Remark. The correspondence between $V_n$ and $R^n$ depends on the choice of a basis $e_1,\ldots,e_n$ in $V_n$. A different basis leads to a different mapping. There are as many different isomorphic mappings between $V_n$ and $R^n$ as there are bases in $V_n$!

## 1.5. Linear equations.

The question whether a vector $b \in R^n$ is a linear combination of vectors $a_1,\ldots,a_m$ out of $R^n$ leads to a system of n linear equations in m unknowns. The question is whether there are scalars $\xi_1,\ldots,\xi_m$ such that

$$a_1\xi_1 + \ldots + a_m\xi_m = b$$

(We prefer now to write the scalar factors $\xi_i$ to the right of the vectors $a_i$.)

Let $a_j$, $j=1,\ldots,m$, and b be represented in terms of coordinates with respect to the natural basis:

$$a_j = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \dots \\ \dots \\ \alpha_{nj} \end{bmatrix} \qquad b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \dots \\ \beta_n \end{bmatrix}$$

Then the following set of equations must hold:

$$\begin{aligned} \alpha_{11}\xi_1 + \dots + \alpha_{1m}\xi_m &= \beta_1 \\ \alpha_{21}\xi_1 + \dots + \alpha_{2m}\xi_m &= \beta_2 \\ \dots\dots\dots\dots\dots\dots & \\ \dots\dots\dots\dots\dots\dots & \\ \alpha_{n1}\xi_1 + \dots + \alpha_{nm}\xi_m &= \beta_n \end{aligned}$$

If b equals the zero vector, the system is called homogeneous. Otherwise it is called inhomogeneous. If a homogeneous system has only the zero solution $\xi_j=0$, $j=1,\dots,m$, then the vectors $a_j$ are linearly independent.

A linear system may also be viewed as a system of equations for forms: Find values for the unknowns $\xi_j$ such that the forms

$$\alpha_{i1}\xi_1 + \dots + \alpha_{im}\xi_m, \quad i=1,\dots,n$$

evaluated for these $\xi_j$ give the numbers $\beta_i$. Forms can be viewed as vectors. The above forms are represented by vectors

$$a^i = (\alpha_{i1},\ldots,\alpha_{im})$$

Also an equation

$$\alpha_{i1}\xi_1 + \ldots + \alpha_{im}\xi_m = \beta_i$$

can be put in correspondence with a vector, namely with the m+1 dimensional vector

$$(a^i,\beta_i) = (\alpha_{i1},\ldots,\alpha_{im},\beta_i)$$

One may start to form linear combinations of these vectors which result in very simple vectors (equations). This is the idea behind the familiar elimination procedures. The final stage of the Gauss-Jordan elimination procedure looks as follows.

$$
\begin{aligned}
\xi_1 \qquad & + \alpha^r_{1,r+1}\xi_{r+1} + \ldots + \alpha^r_{1,m}\xi_m = \beta^r_1 \\
\xi_2 \qquad & + \alpha^r_{2,r+1}\xi_{r+1} + \ldots + \alpha^r_{2,m}\xi_m = \beta^r_2 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\xi_r \; & + \alpha^r_{r,r+1}\xi_{r+1} + \ldots + \alpha^r_{r,m}\xi_m = \beta^r_r \\
& 0 = \beta^r_{r+1} \\
& 0 = 0 \\
& \ldots\ldots \\
& \ldots\ldots \\
& 0 = 0
\end{aligned}
$$

The vectors (equations) of the final stage are linear combinations of those of the initial system. However the converse is also true because any step during the Gauss-Jordan algorithm is reversible. Hence the equations of the initial and the last stage span the same space. The systems of equations are equivalent.

Remark. It may not always be possible to obtain a final stage of equations in which the first r unknowns $\xi_1,\ldots,\xi_r$ are isolated as shown above. A reordering of equations and/or unknowns may be necessary in order to ensure the validity of the above final system.

We introduce the matrix of the homogeneous system. It is a rectangular array A of elements $\alpha_{ij}$

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \dots & \alpha_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \dots & \alpha_{nm} \end{bmatrix}$$

We also introduce the "augmented" matrix of the inhomogeneous system

$$(A,b) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} & \beta_1 \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} & \beta_2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nm} & \beta_n \end{bmatrix}$$

A matrix can be seen as a collection of "row vectors", and, alternatively, as a collection of "column vectors". We shall shortly talk of "rows" and "columns" of a matrix. The following facts are easily deduced from the structure of the GaussJordan reduced system.

(1) The matrix A has r linearly independent rows.

(2) The matrix A has r linearly independent columns. It is seen that the number of linearly independent rows and that of linearly independent columns coincide. This number is called the rank of A. It is denoted

$$r = \text{rank}(A)$$

(3) The solutions $x=(\xi_1,\ldots,\xi_m)$ of the homogeneous system form an $m - r$ dimensional vector subspace of $V_m$. A basis is provided by the columns of the following matrix.

$$\begin{bmatrix} \alpha^r_{1,r+1} & \alpha^r_{1,r+2} & \cdots & \alpha^r_{1,m} \\ \alpha^r_{2,r+1} & \alpha^r_{2,r+2} & \cdots & \alpha^r_{2,m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha^r_{r,r+1} & \alpha^r_{r,r+2} & \cdots & \alpha^r_{r,m} \\ -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

(4) If

$$\beta^r_{r+1} = 0$$

then the rank of the augmented matrix equals

$$\text{rank}(A,b) = \text{rank}(A) = r$$

In this case the linear system is consistent. It has a solution. A particular solution is provided by

$$\begin{aligned}
\xi_1 &= \beta_1^r \\
\xi_2 &= \beta_2^r \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
\xi_r &= \beta_r^r \\
\xi_{r+1} &= 0 \\
\xi_{r+2} &= 0 \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
\xi_n &= 0
\end{aligned}$$

(5) If $\beta_{r+1}^r \neq 0$, then

$$\text{rank}(A,b) = \text{rank}(A) + 1 = r+1$$

In this case the (inhomogeneous) system is inconsistent. It has no solution.

(6) The general solution of a consistent inhomogeneous system is obtained as the

sum of the general solution of the homogeneous system and a particular solution of the inhomogeneous system.

(7) If $m = r$, the solution is unique if it exists.

(8) If $m = n = r$ the solution always exists and is unique. A is an n by n matrix of rank n. Such a matrix is called regular.

## 2. Linear operators.

### 2.1. Definitions.

A linear operator $\Lambda$ is a mapping between vector spaces V and W. It maps any $x \in V$ uniquely onto a $y \in W$. In symbols:

$$y = \Lambda(x)$$

Not every vector $y \in W$ must be the image of a vector $x \in V$. Therefore one says that $\Lambda$ maps V into W. Should the images of all vectors x in V really cover the whole space W, and should one wish to emphasize this fact, one says that $\Lambda$ maps V onto W.

The images of two different vectors $x_1, x_2 \in V$ may coincide in W. Thus the "pre-images" of a vector $y \in W$ need not be unique in V. An important subclass of linear operators will have unique pre-images.

The fundamental property of linearity of the operator $\Lambda$ is expressed by the following equation:

$$\Lambda(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \Lambda(x_1) + \lambda_2 \Lambda(x_2)$$

Thus a linear operator maps a linear combination of vectors onto the linear combination of the individual image vectors in a way that the scalars are preserved.

## 2.2. Examples of linear operators.

### 2.2.1. The linear equations

$$\eta_1 = \alpha_{11}\xi_1 + \ldots + \alpha_{1m}\xi_m$$
$$\eta_2 = \alpha_{21}\xi_1 + \ldots + \alpha_{2m}\xi_m$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\eta_n = \alpha_{n1}\xi_1 + \ldots + \alpha_{nm}\xi_m$$

define a linear operator mapping $R^m$ into $R^n$. Thus we have obtained another important interpretation of a linear system of equations.

2.2.2. Taking the derivative of a polynomial defines a linear operator mapping the space of polynomials onto itself. The subspace of polynomials of degree $\leq n$ is mapped into itself. The space of images is that one of polynomials having degree $\leq n-1$.

2.2.3. Interpolating a continuous function at $n+1$ distinct locations $\xi_0, \xi_1, \ldots, \xi_n$ by a polynomial of degree $\leq n$, defines a linear operator from the space of continuous functions onto the $n+1$ dimensional space of polynomials of degree $\leq n$.

2.2.4. A linear operator mapping a vector space V into R, the set of real numbers, is called a linear functional. The zero functional assigns zero to any vector out of V. All other linear functionals map V onto R. Examples of linear functionals follow.

2.2.5. A linear form

$$\alpha(x) = \alpha_1\xi_1 + \ldots + \alpha_m\xi_m$$

is a linear functional defined on $R^m$.

2.2.6. Evaluating a continuous function at a fixed location $\xi$ defines a linear functional on the space of continuous functions.

2.3. Matrix representation of linear operators.

Let $\Lambda$ be a linear operator mapping the m-dimensional vector space $V_m$ into the n-dimensional space $V_n$. Choose a basis $e_1,\ldots,e_m$ in $V_m$ and a basis $f_1,\ldots,f_n$ in $V_n$. Represent

$$x = \sum_{j=1}^{m} \xi_j e_j$$

$$y = \sum_{j=1}^{n} \eta_j f_j$$

The image $\Lambda(e_j)$ of the basis vector $e_j$ is a vector in $V_n$. Let its representation in terms of the basis $f_1,\ldots,f_n$ be

$$\Lambda(e_j) = \sum_{i=1}^{n} \alpha_{ij} f_i, \quad j=1,\ldots,m$$

We now expand

$$y = \Lambda(x) = \Lambda\left\{ \sum_{j=1}^{m} \xi_j e_j \right\} = \sum_{j=1}^{m} \xi_j \Lambda(e_j) =$$

$$= \sum_{j=1}^{m} \xi_j \sum_{i=1}^{n} \alpha_{ij} f_i = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} \alpha_{ij} \xi_j \right\} f_i$$

Comparing with

$$y = \sum_{i=1}^{n} \eta_i f_i$$

and recalling the uniqueness of coordinates, we obtain

$$\eta_i = \sum_{j=1}^{m} \alpha_{ij} \xi_j$$

We see that the linear operator $\Lambda$ is represented by a matrix

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \dots & \alpha_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \dots & \alpha_{nm} \end{bmatrix}$$

The matrix representation relies on the chosen basis. A different basis leads to a different matrix representation.

2.4. Composition of mappings, matrix product.

Let

$$y = M(x)$$

$$z = \Lambda(y)$$

be two mappings. M maps $V_m$ into $V_n$, and $\Lambda$ maps $V_n$ into $V_p$. The composite mapping $N = \Lambda \circ M$ is defined as

$$z = N(x) = \Lambda \circ M\,(x) = \Lambda(M(x))$$

Let

$$A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \alpha_{p1} & \cdots & \alpha_{pn} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1m} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \beta_{n1} & \cdots & \beta_{nm} \end{bmatrix}$$

be the matrix representation of $\Lambda$ and M, respectively. We are going to find the matrix representation of $N = \Lambda \circ M$, denoted

$$C = \begin{bmatrix} \gamma_{11} & \cdots & \cdots & \gamma_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{p1} & \cdots & \cdots & \gamma_{pm} \end{bmatrix}$$

Substituting

$$\eta_k = \sum_{j=1}^{m} \beta_{kj}\xi_j$$

into

$$\varsigma_i = \sum_{k=1}^{n} \alpha_{ik}\eta_k$$

one obtains

$$\varsigma_i = \sum_{k=1}^{n} \alpha_{ik} \sum_{j=1}^{m} \beta_{kj}\xi_j = \sum_{j=1}^{m} \left\{ \sum_{k=1}^{n} \alpha_{ik}\beta_{kj} \right\} \xi_j$$

It follows that

$$\gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}$$

This leads to the definition of the matrix product

$$C = AB$$

The matrix product is associative, i.e.

$$A(BC) = (AB)C = ABC$$

Associativity follows immediately from the associativity of mappings. It may also be proved directly.

2.5. Inverse operator, inverse matrix.

Let $\Lambda$ be a linear operator mapping $V_n$ onto $W_n$. Let $A=(\alpha_{ij})$ be the matrix representation of $\Lambda$. It follows that the linear system

$$\eta_i = \sum_{j=1}^{n} \alpha_{ij}\xi_j, \quad i=1,\ldots,n$$

has a solution for any choice of $\eta_i$, $i=1,\ldots,n$. From the theory of linear equations it follows that rank(A)=n, and that the solution is necessarily unique. Hence we obtain a mapping $\Lambda^{-1}$ mapping $W_n$ back onto $V_n$. The mapping is necessarily linear. Let $A^{-1}$ be its matrix representation. We call $\Lambda^{-1}$ the inverse operator of $\Lambda$, and $A^{-1}$ the inverse matrix of A. It follows that

$$\Lambda^{-1}\circ\Lambda = I$$
$$A^{-1}A = I$$

Here I denotes in the first case the identity operator mapping $V_n$ identically onto itself: $x = I(x)$. In the second case I denotes the matrix representation of the identity operator. We have

$$I = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{bmatrix}$$

The inverse operator of the inverse $\Lambda^{-1}$ is $\Lambda$ again:

$$\Lambda \circ \Lambda^{-1} = I$$
$$A A^{-1} = I$$

The inverse matrix $A^{-1}$ may be calculated by the Gauss-Jordan procedure. The procedure must be carried out for a general right hand side $\eta_i$, i=1,...,n. (Equivalently, one may apply Gauss-Jordan for n right hand sides represented by the columns of the identity matrix I.)

2.6. Linear functionals.

A linear operator from $V_n$ into R was called a linear functional. Confer example 2.2.4. We write

$$\rho = \lambda(x), \; x \epsilon V_n, \; \rho \epsilon R$$

to indicate that $\lambda$ evaluated at the vector x gives the real number $\rho$.

Example: In two dimensions vectors may be represented by arrows (see section 1.2.9). Linear functionals may then be visualized as systems of equally spaced parallel lines with an orientation. In order to evaluate the functional for a vector, i.e. an arrow, one counts the line spacings between tail and top of the vector. Loosely speaking, one counts how many lines are intersected by the arrow. See fig. 2.1. The sign is taken in agreement with the orientation. The idea generalizes to higher dimensions if systems of parallel hyperplanes are taken instead of systems of lines.

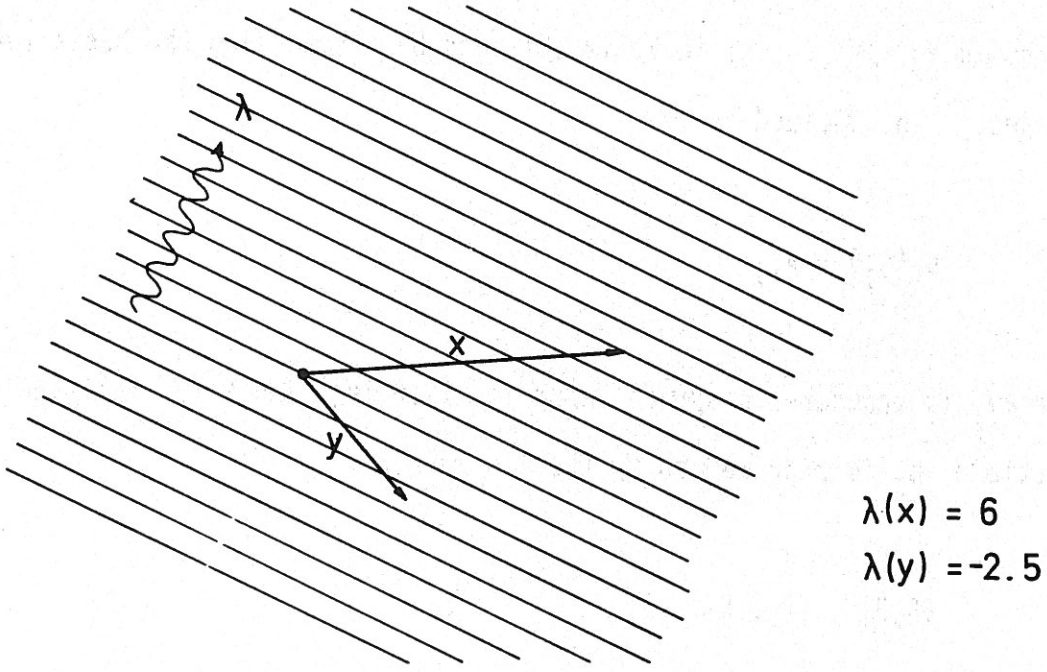



Fig. 2.1. Linear functionals represented by systems of lines

After choosing a basis $e_j$, $j=1,\ldots,n$, in $V_n$, a representation of the functional $\lambda$ by a 1 by n matrix $\underline{\lambda}$ is obtained:

$$\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$$

If a vector x has coordinates $\xi_j$, then the functional $\lambda$ evaluated at x gives the number

$$\lambda(x) = \sum_{i=1}^{n} \lambda_i \xi_i$$

Linear functionals form a vector space. Any linear combination of linear functionals is a linear functional. The vector space of functionals defined on $V_n$ is called the dual vector space. It is frequently denoted $V_n'$. A basis dual to the basis $e_j$, j=1,...,n, in $V_n$ is obtained by introducing the basis functionals $\varepsilon_j$, j=1,...,n, defined by

$$\varepsilon_i(e_j) = \delta_{ij}$$

Here $\delta_{ij}$ is Kronecker's symbol (equaling 1 if i=j, and 0 if i≠j). The basis functional $\varepsilon_j$ is represented by the 1xn matrix

$$(0,0,\ldots,0,1,0,\ldots 0)$$

where the 1 appears at the j-position.

The coordinates of a functional $\lambda$ with respect to the dual basis are precisely the components of its matrix representation.

Example: Fig. 2.2 shows the two arrows representing the chosen basis vectors in

the two-dimensional plane. The dual basis is represented by two systems of lines also shown in the figure.

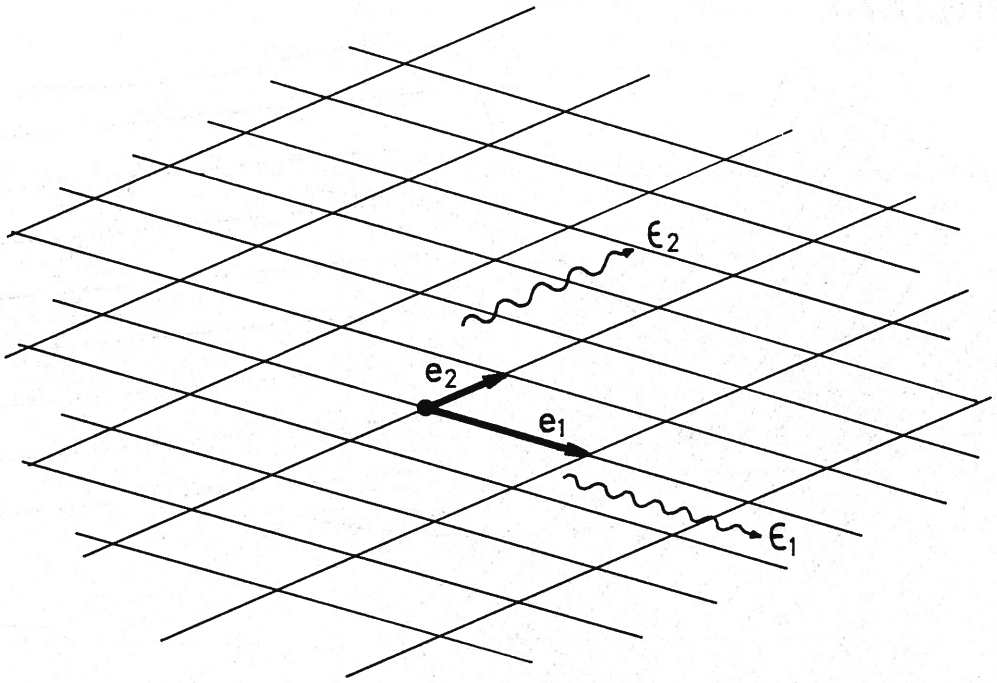



Fig. 2.2. Basis and dual basis.

## 2.7. Coordinates viewed as functionals.

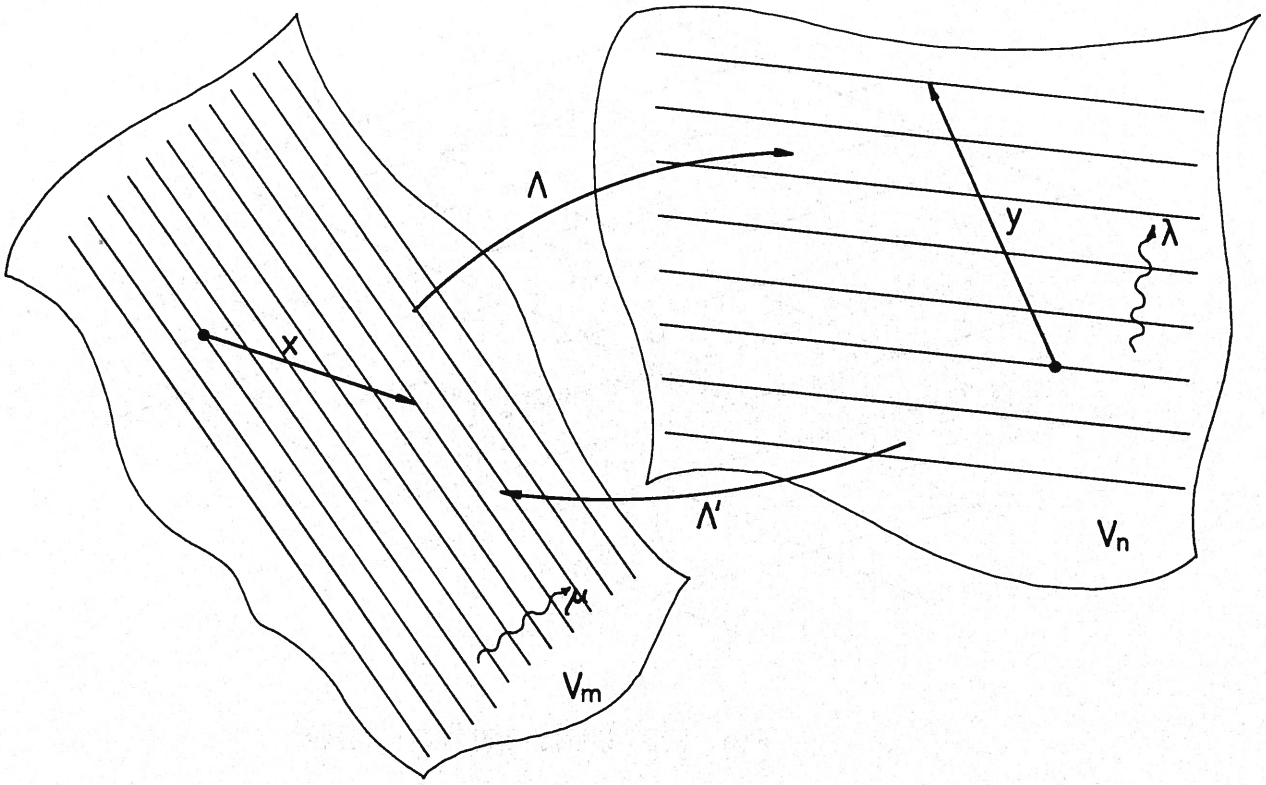
After choosing a basis $e_j$, $j=1,..,n$, in $V_n$, any vector $x$ is represented by its coordinate n-tuple $(\xi_1,...,\xi_n)$. The mapping of $x$ onto its i-th coordinate $\xi_i$ is a linear functional, namely the basis functional $\epsilon_i$.

## 2.8. The dual operator.

Let $\Lambda$ be a linear operator from $V_m$ into $V_n$. Let $\lambda$ be a functional on $V_n$. The equation

$$\mu(x) = \lambda(\Lambda(x))$$

assigns a functional $\mu$ out of $V_m'$ to any $\lambda$ out of $V_n'$. A mapping $\Lambda'$ from $V_n'$ into $V_m'$ is thus obtained. The mapping is linear. $\Lambda'$ is called the dual operator of $\Lambda$ (see fig. 2.3.).




$$\mu(x) = \lambda(\Lambda(x)) = \lambda(y) = 5$$

Fig. 2.3. Illustrating the definition of the dual operator.

Assume that bases in $V_m$, $V_n$, and corresponding dual bases in $V_m'$, $V_n'$ are chosen. Let $A=(\alpha_{ij})$ be the matrix representation of $\Lambda$. We are going to find the matrix representation of $\Lambda'$. We have

$$\mu(x) = \sum_{j=1}^{m} \mu_j \xi_j$$

On the other hand

$$\mu(x) = \lambda(\Lambda(x)) = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} \alpha_{ij}\xi_j = \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} \alpha_{ij}\lambda_i \right\} \xi_j$$

It is seen that

$$\mu_j = \sum_{i=1}^{n} \alpha_{ij}\lambda_i$$

Hence, if $\Lambda$ is represented by the matrix

$$A = \begin{bmatrix} \alpha_{11} & \cdots & \cdots & \alpha_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \cdots & \cdots & \alpha_{nm} \end{bmatrix}$$

then $\Lambda'$ is represented by the transposed matrix

$$A^T = \begin{bmatrix} \alpha_{11} & \cdots & \cdots & \cdots & \alpha_{n1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{1m} & \cdots & \cdots & \cdots & \alpha_{nm} \end{bmatrix}$$

## 3. Matrix calculus.

### 3.1. Preliminaries.

It is time to formalise the computational rules for vectors and matrices. Matrix calculus is the appropriate tool.

An nxm matrix A is a rectangular array of real numbers $\alpha_{ij}$, called its elements:

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \cdots & \alpha_{nm} \end{bmatrix}$$

It will be convenient to identify vectors with their coordinate n-tuples. This is legal in view of the isomorphism between any n-dimensional vector space $V_n$ and $R^n$, the space of n-tuples. It will further be convenient to identify coordinate n-tuples with nx1 matrices, calling them "column vectors". Alternatively, a coordinate n-tuple may be identified with a 1xn matrix and called "row-vector".

As already pointed out in section 1.5, a matrix may be thought of being composed of row vectors, or, alternatively, of column vectors.

### 3.2 Interpretation of a matrix-vector product.

The matrix product of an nxm matrix A and an mx1 column vector gives as result

an nx1 column vector:

$$y = Ax$$

In conventional notation this means

$$\eta_i = \sum_{j=1}^{m} \alpha_{ij}\xi_j$$

The following different interpretations can be given to this system of equations.

(1) A system of linear equations. In section 1.5 the quantities $\eta_i$ were denoted $\beta_i$, $i=1,\ldots,n$.

(2) The vector y is a linear combination of the columns of A. The scalar factors are given by $\xi_j$, $j=1,\ldots,m$.

(3) The n linear functionals, represented by the rows of A, evaluated for the vector x give the results $\eta_i$.

(4) Representation of a linear operator $\Lambda$ from $R^m$ into $R^n$. Confer section 2.3. If natural bases are chosen in $R^m$ and $R^n$, then the images in $R^n$ of the basis vectors in $R^m$ are given by the columns of A:

$$\Lambda(e_j) = \sum_{i=1}^{n} a_{ij} f_i \ , \ j=1,\ldots,m$$

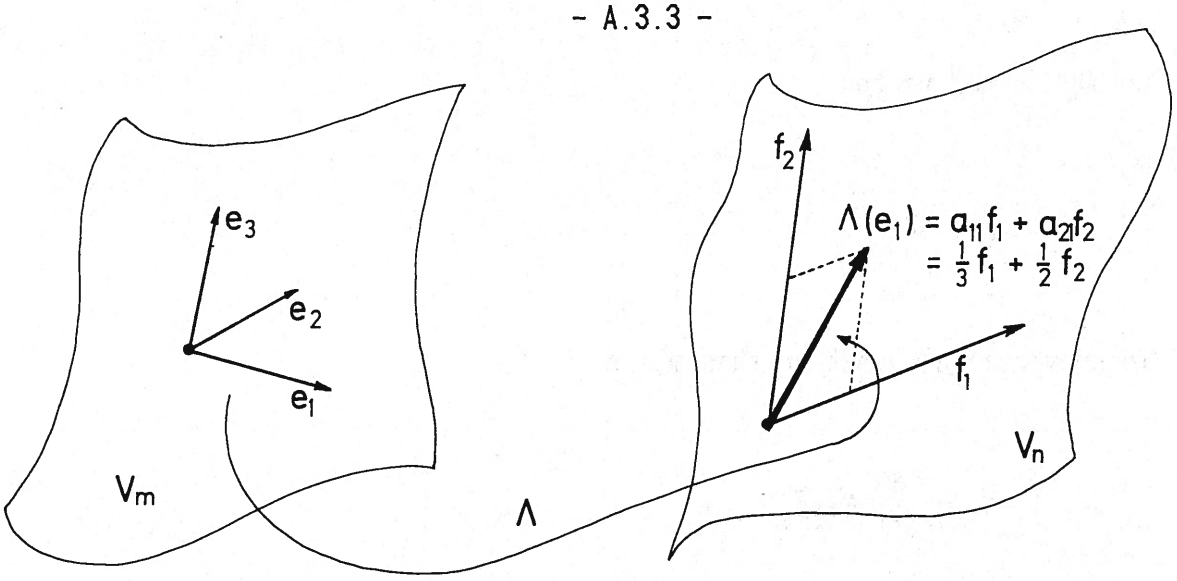



Fig. 3.1. Linear mapping of basis vectors: $\Lambda(e_j) = \sum_{i=1}^{n} a_{ij} f_i$

(5) If A is nxn, the spaces participating in the mapping may be identified. If A is nxn, the mapping is one to one and called an automorphism of $R^n$.

(6) Change of basis in $R^n$, coordinate transformation. Call the natural basis of $R^n$ the old basis. Call the columns of a regular nxn matrix A the new basis. The relation

$$x_{old} = A\, x_{new}$$

may then be viewed as the representation of one and the same vector by coordinates with respect to the old and the new basis. (Note that $x_{new}$ comprises the scalar factors in the linear combination of the vector $x_{old}$ in terms of the new basis.)

The derivation of the above formula runs as follows (see also fig. 3.2!).

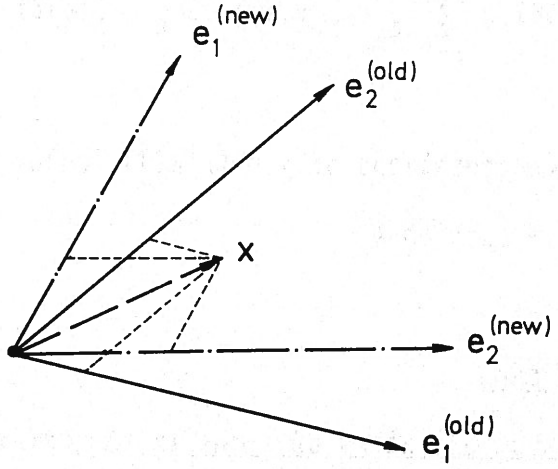



Fig. 3.2. Representing one and the same vector in terms of an old and a new basis.

$$x = \sum_{i=1}^{n} x_i^{(old)} e_i^{(old)} = \sum_{j=1}^{n} x_j^{(new)} e_j^{(new)}$$

Substituting

$$e_j^{(new)} = \sum_{i=1}^{n} a_{ij} e_i^{(old)}$$

one obtains

$$\sum_{i=1}^{n} x_i^{(old)} e_i^{(old)} = \sum_{j=1}^{n} x_j^{(new)} \sum_{i=1}^{n} a_{ij} e_i^{(old)} =$$

$$\sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} x_j^{(new)} \right) e_i^{(old)}$$

By the uniqueness of coordinates we deduce

$$x_i^{(old)} = \sum_{j=1}^{n} a_{ij}\, x_j^{(new)}, \quad \text{i.e.} \quad x^{(old)} = A\, x^{(new)}$$

(7) Further interpretations of $y = Ax$ will follow after the definition of an inner product in section 4.

3.3. Matrix algebra.

3.3.1. Scalar multiplication of a matrix. An nxm matrix A may be multiplied by a scalar factor $\lambda$, yielding an nxm matrix B:

$$B = \lambda A$$

The elements of B are given by

$$\beta_{ij} = \lambda \alpha_{ij}$$

3.3.2. Sum of two nxm matrices. The sum is again nxm:

$$C = A + B$$

The elements of C are given by

$$\gamma_{ij} = \alpha_{ij} + \beta_{ij}$$

Remark. It is seen that the set of nxm matrices forms a vector space. However this viewpoint is not very important for our purposes.

3.3.3. Matrix product. It was already defined in section 2.4. Let A be pxn, B be nxm, C be pxm. The equation

$$C = AB$$

means

$$\gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}$$

For the interpretation of a matrix product as representation of a composite mapping see section 2.4. Formally the matrix product arises if a set of linear expressions is substituted into another:

$y = Bx$ substituted into $z = Ay$ gives $z = A(Bx) = (AB)x$

Important computational rules are

$$A(BC) = (AB)C = ABC \quad \ldots \text{ the associative law}$$

The associative law was already introduced in section 2.4. It was tacitly applied in the above substitution rule. We further have the rule:

$$(A+B)C = AC + BC \quad \ldots \text{ first distributive law}$$
$$A(B+C) = AB + AC \quad \ldots \text{ second distributive law}$$

$$\lambda(AB) = (\lambda A)B = A(\lambda B) = \lambda AB = AB\lambda$$

(scalar factors may be "pushed through matrix products".)

Remark: Note that the matrix product is generally not commutative. If C=AB, the product BA may not even be defined. If BA is defined, as for example in the case of nxn matrices A,B, then BA is generally different from AB.

3.3.4. Transposition. Let A be nxm. The transpose $A^T$ of A was already introduced in section 2.8 . It is an mxn matrix having elements

$$\alpha_{ij}^{(T)} = \alpha_{ji},\ i=1,\ldots,m,\ j=1,\ldots,n.$$

The following computational rule applies:

$$(AB)^T = B^T A^T$$

This may either be verified directly. It may also be inferred from the fact that $A^T$ represents the adjoint operator of that one represented by A: Let B represent a mapping from U into V. Let A represent a mapping from V into W. Then AB represents the composite mapping from U into W. Now, $A^T$ represents the mapping from W' into V', $B^T$ represents that one from V' into U'. Here U',V',W' are the dual spaces. Confer section 2.8. It follows without calculation that the composite mapping from W' into U' is represented by $B^TA^T$ (confer fig. 3.3).

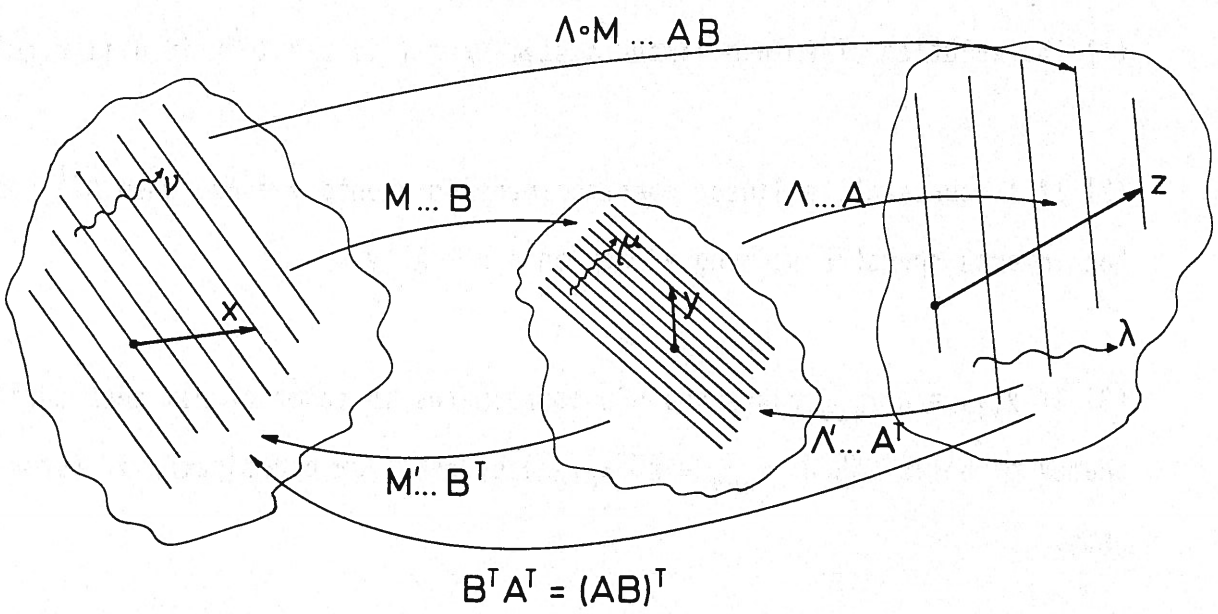



Fig. 3.3. Showing that $(\Lambda\circ M)' = M'\circ\Lambda'$, hence $(AB)^T = B^T A^T$.
(The operators Λ, M are represented by A, B, respectively.)

3.3.5. Inverse matrices. The inverse of a matrix was already introduced in section 2.5. If A is nxn and regular (i.e., rank(A)=n), then the inverse matrix $A^{-1}$ exists and fulfills

$$A^{-1}A = AA^{-1} = I$$

If

$$y = Ax$$

then

$$x = A^{-1}y$$

These equations can be given the following interpretations:

(1) The solution of an nxn linear system $Ax = y$ is $x = A^{-1}y$ if A is regular.

(2) If A represents a linear operator mapping x onto $y = Ax$, then $A^{-1}$ represents the inverse operator mapping y back onto $x = A^{-1}y$.

(3) If $x_{old} = A\, x_{new}$ expresses old coordinates in terms of new ones during a change of basis, then $x_{new} = A^{-1}x_{old}$ expresses new coordinates in terms of old ones.

The following computational rules apply

(1) $(AB)^{-1} = B^{-1}A^{-1}$,

(provided that A,B are nxn and invertible.). The proof relies on the associative law:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

(2) $(A^T)^{-1} = (A^{-1})^T$

The proof follows from transposing $AA^{-1} = I$.

## 4. Inner Products.

### 4.1. Definition.

Let V be a vector space. An inner product assigns a scalar number to any pair of vectors $a,b \in V$. This number is denoted $(a,b)$. The following properties of an inner product are postulated:

$$(a,b) = (b,a) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{ symmetry}$$

$$(\lambda a,b) = \lambda(a,b) \quad \ldots\ldots\ldots\ldots\ldots \text{ homogeneity}$$

$$(a_1+a_2,b) = (a_1,b) + (a_2,b) \quad \ldots\ldots \text{ distributivity}$$

$$(a,a) > 0 \text{ if } a\neq 0 \quad \ldots\ldots\ldots\ldots \text{ positive definiteness}$$

A vector space V equipped with an inner product is called an inner product space. If V is infinite dimensional, one calls it a pre-Hilbert space.

### 4.2. Schwarz's inequality.

It reads

$$(a,b)^2 \leq (a,a)(b,b)$$

Proof: For any scalars $\lambda,\mu$ it follows from positive definiteness that

$$0 \leq (\lambda a+\mu b,\lambda a+\mu b) = \lambda^2(a,a) + 2\lambda\mu(a,b) + \mu^2(b,b)$$

Without loss of generality assume $a\neq 0$. Put $\mu=1$. Then

$$f(\lambda) = \lambda^2(a,a) + 2\lambda(a,b) + (b,b) \geq 0$$

The parabola $f(\lambda)$ must not cross the abscissa. The discriminant must be smaller or equal to zero:

$$(a,b)^2 - (a,a)(b,b) \leq 0$$

This is Schwarz's inequality.

4.3. Norms, distances.

For any vector $a \in V$ the number

$$\|a\| = \sqrt{(a,a)}$$

is meaningfully defined because $(a,a) \geq 0$. The number $\|a\|$ is called the norm or the length of the vector a.

Note that Schwarz's inequality may be rewritten as

$$|(a,b)| \leq \|a\|\|b\|$$

The following properties follow from those of the inner product:

$\|a\| \geq 0$, unless $a=0$ in which case $\|a\| = 0$ ... positivity

$\|\lambda a\| = |\lambda|\|a\|$ ... positive homogeneity

$\|a+b\| \leq \|a\| + \|b\|$ ..... triangle inequality

Proof of the triangle inequality:

$$(a+b,a+b) = (a,a) + 2(a,b) + (b,b)$$

i.e.

$$\|a+b\|^2 = \|a\|^2 + 2(a,b) + \|b\|^2$$

By Schwarz's inequality

$$\|a+b\|^2 \leq \|a\|^2 + 2\,\|a\|\|b\| + \|b\|^2$$

$$\|a+b\|^2 \leq (\|a\| + \|b\|)^2$$

Taking the square root, the triangle inequality is obtained.

The norm allows to define a distance between two vectors:

$$d(a,b) = \|a-b\|$$

The following properties follow immediately from the properties of the norm.

$d(a,b) > 0$ if $a \neq b$, $d(a,a) = 0$ ... positivity

$d(a,c) \leq d(a,b) + d(b,c)$ ... triangle inequality

The definition of a distance makes V a metric space.

Example: In the two dimensional plane, norms and inner products may be defined by a system of concentric and equally spaced circles. See fig. 4.1. If the tail of a vector is placed at the center, the circle passing through its top implies the norm.

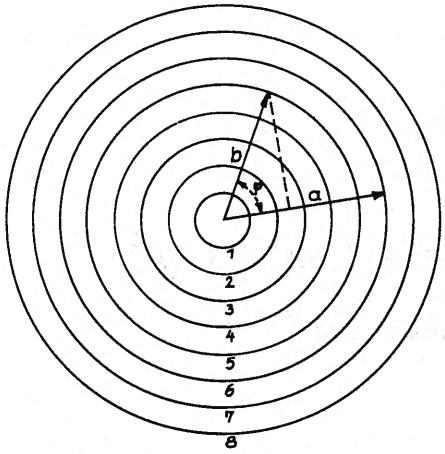



Fig. 4.1. $\|a\| = 6$, $\|b\| = 5$, $\varphi = 60°$, $(a,b) = \|a\|\,\|b\| \cos\varphi = 15$

The inner product can be defined by

$$(a,b) = \|a\|\,\|b\| \cos\varphi$$

This is $\|a\|$ times the norm of the orthogonal projection of b onto a, or likewise, $\|b\|$ times the norm of the orthogonal projection of a onto b. The four properties of the inner product should be verified.

An alternative way to define an inner product in the plane is as follows. The system of circles is changed to a system of ellipses by choosing an arbitrary axis and by shrinking the vertical distances with respect to this axis by an arbitrary factor. Norms and inner products are then defined as shown in fig.

4.2. One can say that norms and inner products of vectors in fig. 4.2 are the norms and inner products of fig. 4.1 applied to the pre-images under the affine mapping that turns circles into ellipses.

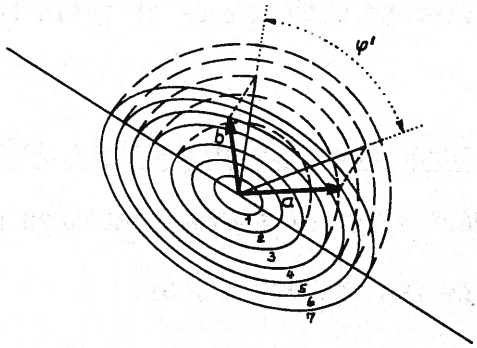



Fig. 4.2. $\|a\| = 6$, $\|b\| = 5$, $(a,b) = \|a\|\,\|b\| \cos \varphi' = 15$

## 4.4. Completeness, Hilbert spaces.

A sequence of vectors $a_1, a_2, \ldots$, is called a Cauchy sequence if for any positive number $\varepsilon$ there exists an index $N(\varepsilon)$ such that

$$d(a_m, a_n) \leq \varepsilon \quad \text{for} \quad n,m \geq N(\varepsilon)$$

A metric space is called complete if any Cauchy sequence possesses a limit element in V: There must be an $a \in V$ such that for any positive $\varepsilon$ there exists $N(\varepsilon)$ such that

$$d(a, a_n) \leq \varepsilon \quad \text{for } n \geq N(\varepsilon)$$

A complete inner product space is called a Hilbert space.

It is not difficult to show (from the completeness of R) that any finite dimensional vector space is complete. It is thus a Hilbert space, although this term is mostly used in context with spaces of infinite dimension.

4.5. Representation of inner products by positive definite matrices.

Let $V_n$ be an inner product space of finite dimension n. Choose a basis $e_1,\ldots,e_n$. Represent the two vectors x,y as

$$x = \sum_i \xi_i e_i, \quad y = \sum_j \eta_j e_j$$

Expand

$$(x,y) = (\sum_i \xi_i e_i, \sum_j \eta_j e_j) = \sum\sum_{ij} \xi_i \eta_j (e_i, e_j)$$

Denote

$$\gamma_{ij} = (e_i, e_j)$$

then the nxn matrix

$$G = \begin{bmatrix} \gamma_{11} & \cdots & \cdots & \cdots & \gamma_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{n1} & \cdots & \cdots & \cdots & \gamma_{nn} \end{bmatrix}$$

is symmetric. We have

$$(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\xi_i\eta_j$$

Identifying x,y with their coordinate n-tuples, interpreting the coordinate n-tuples as nx1 matrices (column vectors), we write

$$(x,y) = x^TGy = y^TGx$$

We see: After choosing a basis, a given inner product is represented by a symmetric matrix. Is the converse also true? Does any symmetric matrix define an inner product? The answer is No! The matrix must fulfill one additional requirement. It must be positive definite.

Definition: A symmetric matrix G is positive definite if for any vector $x \neq 0$ the following inequality holds:

$$x^TGx > 0, \quad \text{if } x \neq 0$$

Equivalent definition: G is positive definite if for any numbers $\xi_1,\ldots,\xi_n$, not all of which are equal to zero, the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\xi_i\xi_j > 0$$

holds.

Positive definiteness is necessary for an inner product. For we must have

$$\|x\|^2 = x^T G x > 0$$

It is also sufficient because one may verify that all other properties of an inner product listed in section 4.1 are fulfilled.

Positive definite matrices are regular. $G^{-1}$ exists. For a proof assume $Gx=0$. Multiply by $x^T$: $x^T G x=0$. This means $\|x\|^2 = 0$. Hence $\|x\| = 0$. Thus $x=0$. We have shown that the homogeneous system $Gx=0$ has only the zero solution. This means that G is regular.

4.6. Orthogonality.

Two vectors x,y are called orthogonal if their inner product vanishes

$$(x,y) = 0$$

Orthogonality depends on the choice of an inner product (but not on the choice of a basis!). If the basis vectors $e_j$, $j=1,\ldots,n$, are orthogonal, we have an orthogonal basis:

$$(e_i,e_i) = \gamma_{ii} > 0$$
$$(e_i,e_j) = 0 \quad \ldots \text{ if } i \neq j$$

If in particular $\|e_i\| = 1$, $i=1,\ldots,n$, we call the basis orthonormal. We then

have

$$(e_i, e_j) = \delta_{ij}$$

The inner product is then represented by the identity matrix:

$$G = I$$

Example: If an orthonormal basis is chosen in the plane, then the inner product of fig. 4.1, i.e.

$$(a,b) = \|a\| \, \|b\| \cos \varphi$$

is represented by the unit matrix:

$$(a,b) = a^T \, I \, b = a^T b$$

Proof: Use polar coordinates, writing

$$a = \begin{bmatrix} r_a \cos \varphi_a \\ r_a \sin \varphi_a \end{bmatrix}, \quad b = \begin{bmatrix} r_b \cos \varphi_b \\ r_b \sin \varphi_b \end{bmatrix}$$

Then

$$a^T b = r_a \, r_b \, \{ \cos \varphi_a \cos \varphi_b + \sin \varphi_a \sin \varphi_b \} =$$

$$r_a\, r_b \cos(\varphi_b - \varphi_a) = \|a\|\,\|b\| \cos\varphi = (a,b)$$

Example: Let the inner product in the plane be given as in fig. 4.2. Referring to fig. 4.2, the following choices of basis vectors imply the representation of the inner product given by the following matrices

$$e_1, e_2 \ldots \quad G = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \quad \gamma_{12} \neq 0$$

$$\bar{e}_1, \bar{e}_2 \ldots \quad G = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_{22} \end{bmatrix}, \quad \gamma_{22} > 1$$

$$\bar{\bar{e}}_1, \bar{\bar{e}}_2 \ldots \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

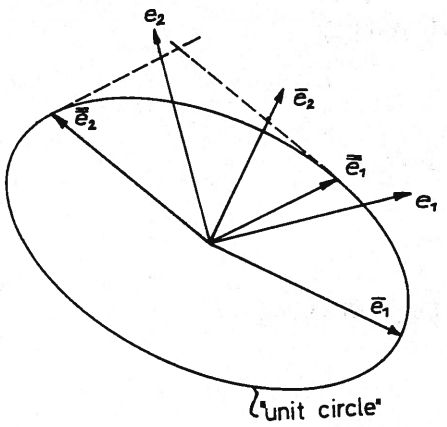



Fig. 4.3.

## 4.7. Gram-Schmid orthogonalization.

The question still remains whether orthogonal bases exist and how to obtain them. A complete answer is given by the Gram-Schmid orthogonalization procedure.

Let V be a vector space. Let $a_1, a_2, \ldots$, be a finite or infinite sequence of vectors. Suppose that any finite subset of these vectors is linearly independent. The orthogonalization procedure derives a sequence of orthonormal vectors $\bar{a}_1, \bar{a}_2, \ldots$, such that

$$\text{span}(a_1, a_2, \ldots, a_m) = \text{span}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m) \text{ for any } m=1,2,\ldots$$

The method proceeds as follows.

Put

$$\bar{a}_1 = \|a_1\|^{-1} a_1$$

Suppose now that the vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_j$ have already been found fulfilling the above specified requirements. Represent the next vector $\bar{a}_{j+1}$ as

$$\bar{a}_{j+1} = \lambda_1 \bar{a}_1 + \lambda_2 \bar{a}_2 + \ldots + \lambda_j \bar{a}_j + \lambda_{j+1} a_{j+1}$$

Require orthogonality of $\bar{a}_{j+1}$ to the earlier obtained vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_j$, i.e., require

$$(\bar{a}_{j+1}, \bar{a}_k) = 0, \quad k=1,\ldots,j$$

This leads to the equations

$$\lambda_k = -(\bar{a}_k, a_{j+1})\lambda_{j+1}$$

Thus we obtain

$$\bar{a}_{j+1} = \left\{ -\sum_{k=1}^{j} (\bar{a}_k, a_{j+1})\bar{a}_k + a_{j+1} \right\} \lambda_{j+1}$$

Abbreviate this as

$$\bar{a}_{j+1} = \lambda_{j+1} h_{j+1}$$

Put

$$\bar{a}_{j+1} = \|h_{j+1}\|^{-1} h_{j+1}$$

4.8. Representation of linear functionals by vectors.

Let x be a fixed vector. Then the inner product

$$(x,y) = \lambda(y)$$

assigns a number to any vector y. All requirements of $\lambda$ to be a linear functional are fulfilled.

It is important that the converse is also true: At least in finite dimensional spaces any linear functional $\lambda$ can be represented by a vector x. (In Hilbert spaces of infinite dimension an additional property of functionals must be required, namely continuity). We adhere to the finite dimensional case. We identify a functional $\lambda$ with the coordinate n-tuple $(\lambda_1, \ldots, \lambda_n)$ with respect to

the dual basis. We know that the $\lambda_i$ are also the elements of the representation of $\lambda$ by an nx1 matrix. We interpret the coordinate n-tuple as a column vector, writing

$$\lambda(y) = \lambda^T y = \sum_{i=1}^{n} \lambda_i \eta_i$$

On the other hand the inner product (x,y) is represented as

$$(x,y) = x^T G y = \sum_{i}\sum_{j} \gamma_{ij} \xi_i \eta_j$$

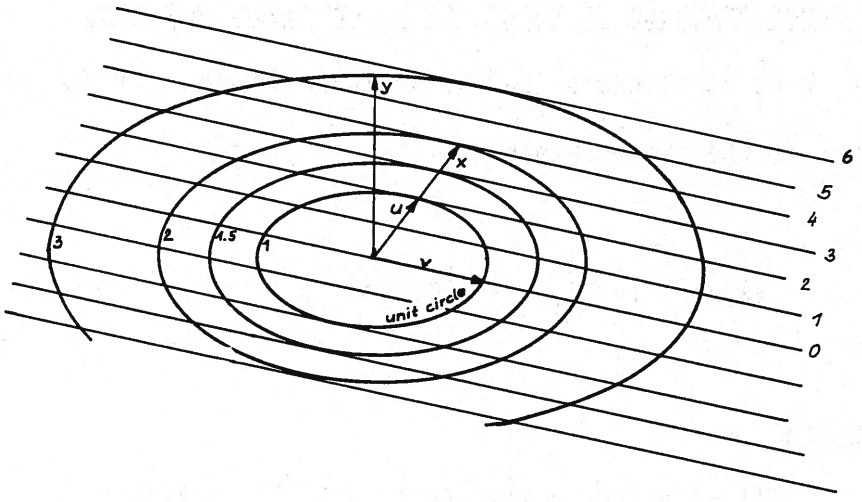



Fig. 4.4. Finding a vector x such that $\lambda(y) = (x,y)$. Choose vectors u, v with $\|u\| = \|v\| = 1$ as shown. Since $\lambda(v) = 0$, x must have the direction of u. $x = \xi u$. Since $\lambda(u) = 2$, we must have $(x,u) = (\xi u,u) = 2$, i.e. $\xi = 2$. This gives the vector x shown in the figure.

Hence the equation

$$\lambda = Gx \text{ or } \lambda_i = \sum_j \gamma_{ij}\xi_j$$

must hold in order to represent the linear functional $\lambda$ by the vector x. Because G is regular, the representation is unique. We obtain

$$x = G^{-1}\lambda$$

The vector x is called the "representer" of the linear functional $\lambda$.

4.9. Inner products of functionals, reproducing kernel.

Let $\lambda$, $\mu$ be functionals. Define an inner product for functionals by their inner product of the representers:

$$(\lambda,\mu) = (x,y), \quad x=G^{-1}\lambda, \; y=G^{-1}\mu$$

We obtain:

$$(\lambda,\mu) = x^T Gy = \lambda^T G^{-1} G G^{-1}\mu = \lambda^T G^{-1}\mu = \lambda^T K\mu$$

It is seen that the inner product $(\lambda,\mu)$ for functionals in V' is represented by the matrix

$$K = G^{-1}$$

This matrix is called the reproducing kernel of the vector space V. The reproducing property of K is described by the equation

$$(K,x) = x$$

More precisely, if $\alpha_{ij}$ are the elements of K, then

$$\sum_{j=1}^{n} \alpha_{ij} \sum_{k=1}^{n} \gamma_{jk} x_k = x_i$$

This is clear from $(K,x) = KGx = Ix = x$.

4.10. The adjoint operator.

If the linear operator $\Lambda$ maps V into W, then the dual operator $\Lambda'$ maps functionals $\lambda \epsilon W'$ back onto functionals $\mu \epsilon V'$. The defining equation of $\Lambda'$ was given in section 2.9. In slightly different notation it reads:

$$\mu(z) = \lambda(\Lambda(z)) = \Lambda'(\lambda)(z), \quad \text{for all } z \epsilon V$$

If $\Lambda$ is represented by the n×m matrix A, then $\Lambda'$ is represented by the m×n matrix $A^T$. The corresponding proof given in section 2.9 becomes very simple if matrix calculus is used. It suffices to rewrite the above equation as

$$\mu^T z = \lambda^T(Az) = (A^T\lambda)^T z, \quad \text{for all } z \epsilon V$$

If inner products are available in V and W, one can define the adjoint operator.

This is done by switching from the functionals $\mu \epsilon V'$, $\lambda \epsilon W'$ to their representers $x \epsilon V$, $y \epsilon W$. According to section 4.8 this is done by means of the relations

$$\mu(z) = (x,z), \quad \text{for all } z \epsilon V$$

and

$$\lambda(z) = (y,z), \quad \text{for all } z \epsilon W$$

The transition from $\lambda$ to $\mu$ via

$$\mu = \Lambda'(\lambda)$$

corresponds to a transition from the representer y of $\lambda$ to the representer x of $\mu$:

$$x = \Lambda^*(y)$$

The operator $\Lambda^*$ is linear because $\Lambda'$ is linear, and because the isometries between V and V' and between W and W' are linear. $\Lambda^*$ is called the adjoint operator.

The defining equation for $\Lambda^*$ is obtained by rewriting the defining equation of $\Lambda'$ as

$$(x,z) = (y,\Lambda(z)) = (\Lambda^*(y),z), \quad \text{for all } z \epsilon V$$

Finally we specify the matrix representation $A^*$ of $\Lambda^*$. Let the inner products in V and W be represented by $G_V$, $G_W$, respectively. Recall that $\mu$ and $\lambda$ are related to their representers x and y by

$$\mu = G_V x, \quad \lambda = G_W y$$

Substitute in

$$\mu = A^T \lambda$$

for $\mu$ and $\lambda$ to obtain

$$G_V x = A^T G_W y$$

or

$$x = G_V^{-1} A^T G_W y$$

It is seen that the matrix representation of $\Lambda^*$ is

$$A^* = G_V^{-1} A^T G_W$$

Remark: The matrix representation $A^*$ of $\Lambda^*$ may also be derived in the following way: Write the defining equation for $\Lambda^*$ as

$$(\Lambda(x), y) = (x, \Lambda^*(y)) \quad \text{for all } x \in V,\ y \in W$$

Use the matrix representations A, $A^*$ to obtain

$$(Ax,y) = (x,A^*y)$$

or

$$x^T A^T G_W y = x^T G_V A^* y,$$

showing once more that $A^* = G_V^{-1} A^T G_W$.

## 5. Projectors.

### 5.1. Decomposition of a vector space into a direct sum of subspaces.

Let V be a vector space, and let $V_A$, $V_B$ be subspaces which have only the zero vector in common. We consider the vector space $V_C$ of all vectors represented as

$$c = a + b, \quad a \epsilon V_A, \; b \epsilon V_B$$

We now show that the above decomposition is unique. Suppose that

$$c = a' + b', \quad a' \epsilon V_A, \; b' \epsilon V_B$$

Subtracting we obtain

$$0 = (a-a') + (b-b')$$

Now $(a-a') \epsilon V_A$, hence $(b-b') = -(a-a') \epsilon V_A$. On the other hand, $(b-b') \epsilon V_B$. It follows that $(b-b')$ is in $V_A$ as well as in $V_B$. It must be the zero vector. Consequently $b=b'$. Similarly $a=a'$ is shown.

Because of the uniqueness of the decomposition, we obtain two mappings, one from $V_C$ onto $V_A$, the other from $V_C$ onto $V_B$:

$$a = \Pi_A(c), \quad b = \Pi_B(c), \quad c \epsilon V_C$$

It is easily seen that these mappings are linear. They are called the

projections of $V_C$ onto $V_A$ and $V_B$ respectively.

Assume that $V_C$ is of finite dimension n. Let the dimension of $V_A$ be m. It follows necessarily that the dimension of $V_B$ is n-m. This is easily proved by choosing a basis in $V_A$ and a basis in $V_B$.

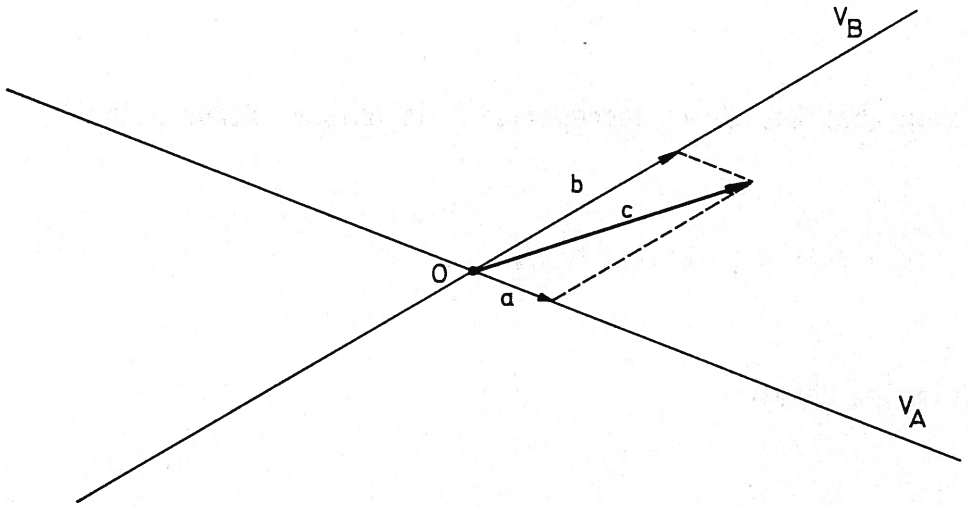



Fig. 5.1. Illustrating the uniqueness of the decomposition
$c = a+b$

5.2. Orthocomplementary subspaces.

Let $V_A$, $V_B$ be subspaces of V. Assume that any vector in $V_A$ is orthogonal to any vector in $V_B$:

$$(a,b) = 0, \text{ if } a \in V_A \text{ and } b \in V_B$$

A vector belonging to $V_A$ as well as to $V_B$ is orthogonal to itself. Its norm is zero; it must be the zero vector. Hence $V_A$ and $V_B$ have only the zero vector in common. We may form the direct sum $V_C$ as we did in the previous subsection. We

call $V_A$ and $V_B$ orthocomplementary subspaces of $V_C$. We also say that $V_B$ is the orthocomplement of $V_A$. Likewise, $V_A$ is the orthocomplement of $V_B$. In symbols

$$V_B = V_A^{\perp}, \quad V_A = V_B^{\perp}$$

Note that the orthocomplement of the orthocomplement gives the original subspace

$$(V_A^{\perp})^{\perp} = V_A$$

The linear operators $\Pi_A$ and $\Pi_B$ introduced in the previous section are called orthogonal projectors.

5.3. The theorem by Pythagoras.

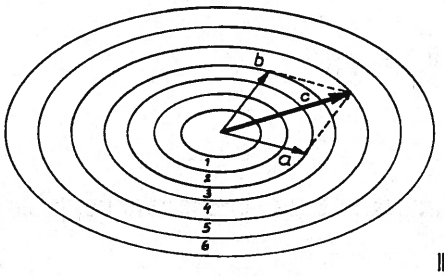
Let a vector $c \epsilon V_C$ be represented as the sum of its orthogonal projections onto orthocomplementary subspaces $V_A$ and $V_B$:

$$c = a + b$$

Then

$$\|c\|^2 = \|a\|^2 + \|b\|^2$$

Proof: $\|c\|^2 = (c,c) = (a+b,a+b) = (a,a) + 2(a,b) + (b,b) = \|a\|^2+\|b\|^2$, because $(a,b)=0$.




$$\|c\|^2 = \|a\|^2 + \|b\|^2$$
$$5^2 = 3^2 + 4^2$$

Fig. 5.2. The theorem by Pythagoras.

Theorem. Consider the following extremum problem: Given $c \in V_C$, find $x \in V_A$ such that

$$\|c-x\| = \text{minimum},$$

The solution is

$$x = a = \Pi_A(c).$$

Proof: Decompose

$$c = a + b,\ a \in V_A,\ b \in V_B$$

Represent

$$c-x = (a-x) + b$$

It is seen that $(a-x) \epsilon V_A$ and $b \epsilon V_B$. The theorem by Pythagoras implies

$$\|c-x\|^2 = \|a-x\|^2 + \|b\|^2$$

This is minimal for a=x, which was to be shown.

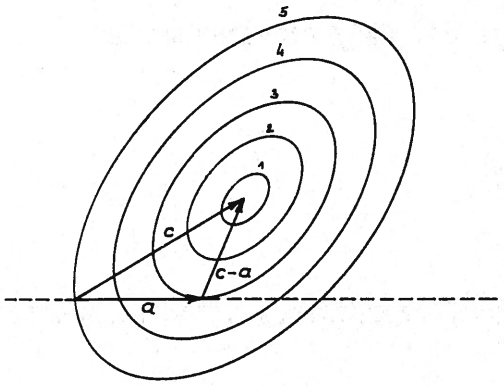



Fig. 5.3. Projection as the solution of a minimum problem

5.4. Matrix representation of orthogonal projectors.

Assume $V_A$, $V_B$ and $V_C$ of dimension m, n-m, n, respectively. Let an inner product in $V_C$ be represented by the symmetric and positive definite matrix G. For simplicity, identify $V_A$, $V_B$, $V_C$ with $R^m$, $R^{n-m}$, $R^n$. Let the columns of the nxm matrix A be a basis in $V_A$. Likewise, let the columns of the nx(n-m) matrix B be a basis in $V_B$. Then the columns of the matrix (A,B), whose columns are those of

A followed by those of B, form a basis in $V_C$. Moreover, the orthogonality requirement of $V_A$ and $V_B$ implies:

$$A^TGB = 0,\ B^TGA = 0$$

(Recall that inner products (a,b) are written $a^TGb$; note that the elements of $A^TGB$ are just all inner products of any basis vector in $V_A$ with any basis vector in $V_B$.)

Vectors $aeV_A$ and $beV_B$ are uniquely represented as

$$a = Ax,\ b = By$$

Here x and y are vectors of coordinates with respect to the bases in $V_A$, $V_B$. The decomposition

$$c = a + b,\ aeV_A,\ beV_B$$

is therefore equivalently written as

$$c = Ax + By$$

Form all inner products of c with basis vectors in A, i.e., multiply the above equation by $A^TG$. In view of $A^TGB = 0$ we obtain

$$(A^TGA)x = A^TGc$$

This is a set of equations called "normal equations". We prove that the mxm normal equation matrix $A^TGA$ is symmetric and positive definite. Symmetry follows from the symmetry of G and the transposition rule for matrix products:

$$(A^TGA)^T = A^TG^T(A^T)^T = A^TGA$$

In order to prove positive definiteness, we must show that for any nonzero x we have $x^T(A^TGA)x > 0$. We note that

$$x^T(A^TGA)x = (Ax)^TG(Ax) = (Ax,Ax) = \|Ax\|^2 \geq 0$$

Assume now that $\|Ax\|^2=0$. Then $\|Ax\|=0$. By the positivity of the norm we infer $Ax=0$. From the uniqueness of coordinates with respect to the basis in $V_A$ it follows that $x=0$. The proof of positive definiteness is complete.

The solution of the normal equations is uniquely obtained as

$$x = (A^TGA)^{-1}A^TGc$$

Inserting this into $a = Ax$, we get

$$a = \Pi_A(c) = A(A^TGA)^{-1}A^TGc = P_Ac$$

Here we have introduced the matrix

$$P_A = A(A^TGA)^{-1}A^TG$$

It is the matrix representation of the operator $\Pi_A$ projecting $V_C$ onto $V_A$. Similarly

$$P_B = B(B^TGB)^{-1}B^TG$$

represents $\Pi_B$, the projection operator from $V_C$ onto $V_B$. The equation

$$c = a + b$$

or

$$c = \Pi_A(c) + \Pi_B(c)$$

shows that

$$\Pi_A + \Pi_B = I,$$

the identity operator in $V_C$. This equation implies the matrix equation

$$P_A + P_B = I$$

which may also be proved algebraically as follows. Multiply the last equation from behind by the matrix (A,B). Obtain $(P_A+P_B)(A,B) = (A,B)$. From the

regularity of (A,B) whose columns form a basis in $V_C$ infer the desired relation $P_A+P_B=I$.

We further note the following relation

$$\Pi_A \circ \Pi_B = 0$$

The corresponding matrix relation is

$$P_A P_B = 0$$

The proof is geometrically as easy as algebraically.

5.5. Projections of functionals.

Due to the isometry between a vector space V and its dual V' (confer section 4.8), projections of functionals can be introduced in a very natural way. We consider the dual spaces $V_A'$, $V_B'$, $V_C'$. If G represents the inner product in $V_C$ with respect to the chosen basis, then $K=G^{-1}$ represents the inner product in $V_C'$ with respect to the dual basis. Recall that any linear functional $\lambda \epsilon V_C'$ is related to its representing vector x by $\lambda=Gx$, $x=K\lambda$. If $V_A$ is spanned by the columns of the matrix A, then $V_A'$ is spanned by the columns of A'=GA. Likewise $V_B'$ is spanned by B'=GB. $V_C'$ is decomposed into orthocomplementary subspaces $V_A', V_B'$. The relation

$$A'^T K B' = 0$$

expresses the orthogonality of $V_A'$, $V_B'$ in matrix form. The projectors $\Pi_A'$, $\Pi_B'$ onto $V_A'$, $V_B'$ are represented by

$$\lambda_A = P_A'\lambda, \quad \lambda_B = P_B'\lambda$$

with

$$P_A' = A'(A'^{T}KA')^{-1}A'^{T}K$$
$$P_B' = B'(B'^{T}KB')^{-1}B'^{T}K$$

This is all very obvious because of the isometry. It is also obvious that the representer $x_A$ of the projected functional $\lambda_A$ is the projection $P_Ax$ of the representer x of $\lambda$. However there is the following interesting characterization of functionals $\alpha \epsilon V_A'$ and $\beta \epsilon V_B'$ in terms of the vector subspaces $V_A$, $V_B$:

$\alpha \epsilon V_A'$ is equivalent to $\alpha(y) = 0$ for any $y \epsilon V_B$

$\beta \epsilon V_B'$ is equivalent to $\beta(x) = 0$ for any $x \epsilon V_A$

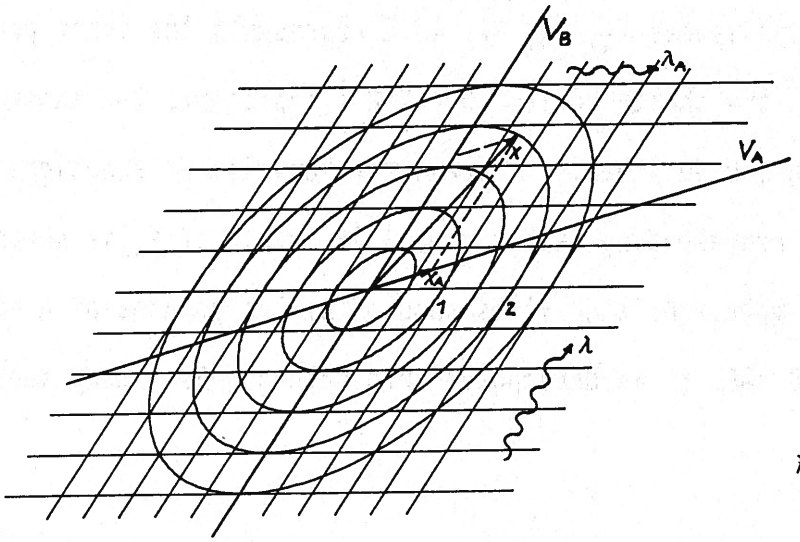



Fig. 5.4. The projection of a functional.

This is clear, because if $\alpha \epsilon V_A'$, $\alpha$ is represented by a vector $a \epsilon V_A$, and $\alpha(y) = (a,y)$. It follows that $\alpha(y) = 0$ for any $y \epsilon V_B$. On the other hand, if a functional $\alpha$ is represented by a, and if $\alpha(y) = (a,y) = 0$ for any $y \epsilon V_B$, then a must be in $V_A$. Hence $\alpha$ must be in $V_A'$.

The projector of a functional $\lambda_A = P_A' \lambda$ is characterized in the following way

$$\lambda_A(y) = \lambda(y) \quad \text{for } y \epsilon V_A$$
$$\lambda_A(y) = 0 \quad \text{for } y \epsilon V_B$$

The proof follows again easily from $\lambda_A(y) = (x_A, y)$.

The projection operator $\Pi_A'$ is the dual of the projection operator $\Pi_A$. Cf. section 2.9.

$$\lambda_A(x) = (\Pi_A' \lambda)(x) = \lambda(\Pi_A(x))$$

For a proof one just inserts for x vectors either in $V_A$ or in $V_B$.

We also have the following

Theorem. If $\lambda$ is a linear functional in $V_C'$, than the projection $\lambda_A = P_A' \lambda$ is the solution of the following extremum problem: Given $\lambda$, find a functional $\mu$ such that

$\|\mu\|$ = minimum

subject to

$\mu(a) = \lambda(a)$ for any $a \epsilon V_A$

Proof: The relation $\mu(a)=\lambda(a)$ for any $a\epsilon V_A$, is equivalent to $(\lambda-\mu)(a) = 0$ for any $a\epsilon V_A$. By our earlier characterization of $V_B'$, this is equivalent to $(\lambda-\mu)\epsilon V_B'$. This in turn is equivalent to $(\lambda-\mu)_A=0$, or $\mu_A=\lambda_A$. We see that any $\mu$ qualifying for minimization is representable as $\mu=\lambda_A+\nu$, $\nu\epsilon V_B'$. By the theorem of Pythagoras we have $\|\mu\|^2 = \|\lambda_A\|^2 + \|\nu\|^2$. Hence $\mu=\lambda_A$ is the smallest.

Suppose that $\mu_A$ is a functional whose domain is $V_A$. Its values for vectors not in $V_A$ are unspecified. We call $\mu$ an extension of $\mu_A$ to $V_C$ if $\mu$ is defined on all of $V_C$ and if $\mu(a) = \mu_A(a)$ for any $a\epsilon V_A$. The following theorem is rather obvious.

Theorem. The extension $\mu$ of $\mu_A$ having the smallest possible norm is given by

$\mu(a) = \mu_A(a)$ ... for $a\epsilon V_A$

$\mu(b) = 0$ ..... for $b\epsilon V_B$

Equivalently

$\mu(x) = \mu_A(\Pi_A(x))$, $x\epsilon V_C$

The minimal norm is given by

$$\|\mu\| = \|\mu_A\|$$

Of course, $\mu_A$ is the projection of $\mu$ onto $V_A'$.

Remark: Referring to section 4.10 on the redefinition of the adjoint operator in case of inner product spaces, note that $P_A$ is a self-adjoint operator:

$$P_A^* = P_A, \quad P_A^* = G^{-1}P_A{}^T G$$

The adjoint equals the original operator.

## 6. Least squares adjustment.

### 6.1. Projecting the vector of observations.

It remains to change notation in order to conform with sacred traditions in least squares adjustment. Let L be the n-dimensional vector space of observations. A vector $\ell \in L$ has coordinates

$$\ell = \begin{bmatrix} \ell_1 \\ \ell_2 \\ .. \\ .. \\ \ell_n \end{bmatrix}$$

Any coordinate corresponds to one individual measurement such as a direction, a distance, an azimuth or a Doppler count. Of course, in an originally nonlinear problem the observations are replaced by small increments with respect to approximative quantities. Note that we deviate from the earlier rule to use Greek letters for coordinates.

We introduce the subspace $L_A$ of adjusted observations. Corrections v must be added to the observations in order to force the adjusted observations into $L_A$:

$$\ell + v \in L_A$$

Equivalently

$$\ell + v = Ax$$

As before, the columns of the nxm matrix A are a basis of $L_A$. The vector v of corrections is a member of L. The size of any vector in L is measured by a norm derived from an inner product. Let the inner product be represented by the "weight matrix" P. The matrix P is symmetric and positive definite. It need not be diagonal, although in most applications it is assumed to be so. We like to have corrections as small as possible. Thus we arrive at

$$\|\ell - a\| = \text{minimum}, \quad a \in L_A, \text{ i.e., } a = Ax$$

The solution was already obtained in the previous section:

Form the normal equations

$$(A^T P A)x = A^T P \ell$$

to obtain

$$\ell + v = Ax = A(A^T P A)^{-1} A^T P \ell = P_A \ell$$

The corrections are given by

$$v = -(\ell - P_A \ell) = -(I - P_A)\ell = -P_B \ell$$

The requirement $\ell + v \in L_A$ is equivalently replaced by $\ell + v$ orthogonal to $L_B$, i.e.

$$B^T P(\ell + v) = 0$$

In conditioned adjustment it is customary to replace the columns of the matrix B by the corresponding functionals. Confer section 4.8. Thus one introduces

$$B' = PB, \quad B = P^{-1}B'$$

Inserting into the previous equation gives the condition equations

$$B'^{T}(\ell+v) = 0$$

Minimizing v gives, as we know, the solution

$$v = -P_B\ell$$

$$v = -B(B^{T}PB)^{-1}B^{T}P\ell = -P^{-1}B'(B'^{T}P^{-1}B')^{-1}B'^{T}\ell$$

One introduces correlates k by

$$k = -(B'^{T}P^{-1}B')^{-1}B'^{T}\ell$$

This permits us to write

$$v = P^{-1}B'k$$

The correlates are the solution of the normal equations of adjustment by condition:

$$(B'^{T}p^{-1}B')k + w = 0$$

The vector w of discrepancies is given by

$$w = B'^{T}\ell$$

6.2 Inhomogeneous form of least squares adjustment.

Frequently an adjustment problem is posed as follows.

Minimize $\|v\|^2$ subject to

$$\ell+v = a_0 + Ax \quad \text{(variation of parameters)}$$

or subject to

$$B^{T}(\ell+v) = b_0 \quad \text{(conditions)}$$

In the case of variation of parameters the requirement is

$$\|\ell-(a_0+Ax)\|^2 = \text{minimum}$$

If it is rewritten as

$$\|(\ell-a_0) - Ax\|^2 = \text{minimum}$$

we arrive at the earlier case with $\ell$ replaced by $\ell-a_0$.

The solution is obtained from the normal equations

$$(A^T PA)x = A^T P(\ell - a_0)$$

$$v = -(I - P_A)(\ell - a_0)$$

In case of the conditioned adjustment we introduce a particular solution $a_0$ of the inhomogeneous system $B'^T a_0 = b_0$. We then have

$$B'^T((\ell - a_0) + v) = 0$$

This reduces again to the earlier case if $\ell$ is replaced by $(\ell - a_0)$. The solution is

$$v = -P_B(\ell - a_0) = PB'k$$

with

$$k = -(B'^T P^{-1} B')w$$

and

$$w = B'^T(\ell - a_0) = B'^T \ell - b_0$$

## 6.3. The fundamental rectangular triangle of least squares adjustment.

The triangle is formed by

| | |
|---|---|
| hypothenuse | $c = \ell - a_0$ |
| 1st short side | $a = Ax = \Pi_A(\ell - a_0)$ |
| 2nd short side | $b = -v = \Pi_B(\ell - a_0) = (I - P_A)(\ell - a_0)$ |

The vector $(\ell-a_0)$ is orthocomplementary decomposed into a=Ax and -v. The theorem by Pythagoras shows:

$$\|\ell-a_0\|^2 = \|Ax\|^2 + \|v\|^2$$

or

$$\|v\|^2 = \|\ell-a_0\|^2 - \|Ax\|^2$$

or

$$v^TPv = (\ell-a_0)^TP(\ell-a_0) - x^TA^TPAx$$

Putting a=Ax, one also recognizes $(a,a) = (a,\ell-a_0+v) = (a,\ell-a_0)$, because $(a,v) = 0$. Hence

$$v^TPv = (\ell-a_0)^TP(\ell-a_0) - (\ell-a_0)^TPAx$$

Furthermore $(v,v) = -(v,\ell-a_0)$. Using $v = P^{-1}B'k$, one gets

$$v^TPv = -k^TB'^TP^{-1}P(\ell-a_0) = -k^T(B'^T\ell-b_0) = -k^Tw$$

6.4. Least squares adjustment by projecting functionals.

Let L' be the dual space of L, the space of observations. Any $\lambda \in L'$ is a linear form in the observables. Thus if the observations are angles, distances e.t.c., then $\lambda$ may refer to a coordinate of a station, to an area, or to any other quantity depending linearly on the observations. (Nonlinear adjustment problems have to be linearized, of course.) Recall that the coordinates

of any vector are functionals too. Hence $\lambda$ may just refer to any particular coordinate $\ell_i$ of $\ell$.

The size of $\lambda$ is measured by its norm $\|\lambda\|$. We have

$$\|\lambda\|^2 = \lambda^T Q \lambda$$

Here $Q=P^{-1}$, the matrix representation of the reproducing kernel of L.

The subspace $L_A$ is the space of adjusted observations. We want to replace $\lambda$ by a functional $\lambda_A$ such that $\lambda_A(a)$ coincides with $\lambda(a)$ for any adjusted observation $a \in L_A$, and such that $\lambda_A$ is as small as possible. The minimum problem

$$\|\mu\| = \text{minimum}$$

subject to

$$\mu(a) = \lambda(a) \text{ for any } a \in L_A$$

was solved in section 5.5. The solution is the projection

$$\lambda_A = \Pi_A'(\lambda)$$

If $\lambda$ is identified with its coordinate column vector $\lambda$ (with respect to the dual basis), then

$$\lambda_A = P_A' \lambda$$

with

$$P'_A = A'(A'^T Q A')^{-1} A'^T Q$$

Putting A'=PA, we obtain

$$\lambda_A = PA(A^T PA)^{-1} A^T \lambda$$

Applying the adjusted functional toward the observation vector, we get

$$\lambda_A(\ell) = \lambda_A^T \ell = \lambda^T A(A^T PA)^{-1})A^T P\ell = \lambda^T P_A \ell = \lambda^T(\ell_A)$$

The familiar rule is recovered. The adjusted linear functional applied to the original observations is obtained by inserting the adjusted observations into the original functional. This demonstrates the equivalence of the two adjustment procedures.

Remark. The functional approach toward least squares adjustment has the following advantages:

(1) It lends itself to a statistical interpretation. Confer part B.

(2) It generalizes to vectors of infinitely many observations. It may not be possible to assign a finite norm to such a vector of observations. Hence least squares adjustment of stochastic processes relies on the functional approach.

## 7. Partitioned matrices.

### 7.1. Definitions.

Consider a matrix of size $(n_1+ n_2)x(m_1+ m_2)$:

$$A = \left[\begin{array}{ccc:ccc} \alpha_{1,1} & \cdots & \alpha_{1,m_1} & \alpha_{1,m_1+1} & \cdots & \alpha_{1,m_1+m_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n_1,1} & \cdots & \alpha_{n_1,m_1} & \alpha_{n_1,m_1+1} & \cdots & \alpha_{n_1,m_1+m_2} \\ \hdashline \alpha_{n_1+1,1} & \cdots & \alpha_{n_1+1,m_1} & \alpha_{n_1+1,m_1+1} & \cdots & \alpha_{n_1+1,m_1+m_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n_1+n_2,1} & \cdots & \alpha_{n_1+n_2,m_1} & \alpha_{n_1+n_2,m_1+1} & \cdots & \alpha_{n_1+n_2,m_1+m_2} \end{array}\right]$$

The matrix may be partitioned as indicated by the dashed lines. Calling the submatrices $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$, one writes

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

The concept of partitioning may be generalized in an obvious way:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix}$$

The submatrices are sometimes called "blocks". Also vectors may be partitioned into subvectors.

$$x = \left[\begin{array}{c} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \xi_m \\ \hline \xi_{m+1} \\ \cdot \\ \cdot \\ \xi_n \end{array}\right] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

7.2. Computational rules.

7.2.1. Scalar multiplication. Partitioned matrices may be multiplied by a scalar. Obviously all submatrices are multiplied by the scalar.

7.2.2. Addition. Partitioned matrices may be added. Provided that the dimensions of corresponding submatrices coincide, one simply adds corresponding submatrices.

7.2.3. Transposition. The transpose of a partitioned matrix may be formed, e.g.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad A^T = \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}$$

7.2.4. Matrix multiplication. Partitioned matrices may be multiplied under suitable circumstances. This is best explained by an example. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{bmatrix}$$

We say that A has p=3 generalized rows and n=2 generalized columns. The number of generalized columns of A coincides with the number of generalized rows of B. B has m=4 generalized columns. Let the matrix C be composed of p×m blocks:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \end{bmatrix}$$

Then the product

$$C = AB$$

may be formed by

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

provided that all matrix products $A_{ik}B_{kj}$ are defined. This is the case if in any

block multiplication $A_{ik}B_{kj}$ the first factor has as many (ordinary) columns as the second factor has (ordinary) rows.

7.3 Block diagonality.

The n×n matrix A is called block-diagonal if A is represented as

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

Here $A_{11}$ is $n_1 \times n_1$ and $A_{22}$ is $n_2 \times n_2$ and $n_1+n_2=n$. $A_{11}$ and $A_{22}$ are square matrices.

If the inverse matrices $A_{11}^{-1}$, $A_{22}^{-1}$ exist, then

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$$

This is readily proved by verifying

$$AA^{-1} = \begin{bmatrix} A_{11}A_{11}^{-1} & 0 \\ 0 & A_{22}A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

## 7.4 Block-Gauss-elimination.

Consider a linear system

$$Ax = b$$

Let it be consistently partitioned as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Let $A_{11}$, $A_{22}$ be square matrices

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right] = \left[\begin{array}{c} b_1 \\ \hline b_2 \end{array}\right]$$

According to the computational rules for partition we may write

$$A_{11}x_1 + A_{12}x_2 = b_1$$
$$A_{21}x_1 + A_{22}x_2 = b_2$$

This looks very similar to 2 equations in 2 unknowns. Let us apply the familiar elimination procedure. We assume that $A_{11}^{-1}$ exists. We premultiply the first equation by $A_{21}A_{11}^{-1}$ and subtract from the $2^{nd}$. The result is

$$A_{11}x_1 + A_{12}x_2 = b_1$$
$$0 + (A_{22}-A_{21}A_{11}^{-1}A_{12})x_2 = b_2 - A_{21}A_{11}^{-1}b_1$$

We assume that we may uniquely solve the second set of equations:

$$x_2 = (A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1}(b_2-A_{21}A_{11}^{-1}b_1)$$

We substitute back into the first equation obtaining:

$$A_{11}x_1 = b_1 - A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1}(b_2-A_{21}A_{11}^{-1}b_1)$$
$$x_1 = [A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}]b_1$$
$$- A_{11}^{-1}A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1}b_2$$

Abbreviating

$$A_{11}^{(-1)} = A_{11}^{-1} + A_{11}^{-1}A_{12}A_{22}^{(-1)}A_{21}A_{11}^{-1}$$
$$A_{12}^{(-1)} = -A_{11}^{-1}A_{12}A_{22}^{(-1)}$$
$$A_{21}^{(-1)} = -A_{22}^{(-1)}A_{21}A_{11}^{-1}$$
$$A_{22}^{(-1)} = (A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1}$$

we get

$$x_1 = A_{11}^{(-1)}b_1 + A_{12}^{(-1)}b_2$$
$$x_2 = A_{21}^{(-1)}b_1 + A_{22}^{(-1)}b_2$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11}^{(-1)} & A_{12}^{(-1)} \\ A_{21}^{(-1)} & A_{22}^{(-1)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

or

$$x = A^{-1}b$$

we have recovered the block-decomposition of the inverse!

7.5 Theoretical background of partitioned matrices.

Let V be a vector space of dimension n and let $V_1$ and $V_2$ be vector subspaces of dimension $n_1$, $n_2$ with $n = n_1 + n_2$. Assume that the zero vector is the only vector common to $V_1$ and $V_2$. As explained in section 5.1, V is the direct sum of $V_1$ and $V_2$. The subspaces $V_1$ and $V_2$ are not necessarily orthocomplementary.

Any vector $x \in V$ is uniquely decomposed as

$$x = x_1 + x_2, \quad x_1 \in V_1, \quad x_2 \in V_2$$

As pointed out in section 5.1, the two mappings $x \to x_1$ and $x \to x_2$ are linear. We write

$$x_1 = \Pi_1(x)$$
$$x_2 = \Pi_2(x)$$

The $\Pi_1$, $\Pi_2$ are called projection operators. They are not necessarily orthogonal projectors. Let $V_m$ and $V_n$ be vector spaces. Let

$V_m = V_{m_1} + V_{m_2}$, $V_m$ is the direct sum of $V_{m_1}$ and $V_{m_2}$

$V_n = V_{n_1} + V_{n_2}$, $V_n$ is the direct sum of $V_{n_1}$ and $V_{n_2}$

Let $\Lambda$ be a linear operator $V_m \to V_n$. Let

$$y = \Lambda(x)$$

Decompose

$$x = x_1 + x_2, \qquad x_1 \in V_{m_1},\ x_2 \in V_{m_2}$$
$$y = y_1 + y_2, \qquad y_1 \in V_{n_1},\ y_2 \in V_{n_2}$$

We have

$$y = \Lambda(x)$$

Let $\Pi_1$, $\Pi_2$ be $x \to x_1$, $x \to x_2$

and $R_1$, $R_2$ be $y \to y_1$, $y \to y_2$

Then

$$y_1 = R_1 y = R_1 \circ \Lambda(x) = R_1 \circ \Lambda(\Pi_1(x) + \Pi_2(x))$$

i.e.

$$y_1 = \Lambda_{11}(x_1) + \Lambda_{12}(x_2)$$

similarly:

$$y_2 = \Lambda_{21}(x_1) + \Lambda_{22}(x_2)$$

Note that $\Lambda_{ij}$ maps $V_{m_j}$ into $V_{n_i}$. Choose bases as follows

$$
\begin{aligned}
V_{m_1} &: e_1 \quad , \ldots , e_{m_1} \\
V_{m_2} &: e_{m_1+1} , \ldots , e_m \\
V_{n_1} &: f_1 \quad , \ldots , f_{n_1} \\
V_{n_2} &: f_{n_1+1} , \ldots , f_n
\end{aligned}
$$

Let A be the matrix representation of $\Lambda$. Partition

$$
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
$$

It is easily verified that $A_{ij}$ are the representations of $\Lambda_{ij}$. Thus it is seen that the calculus of partitioned matrices relies on two assumptions:

(1) Decomposition of the participating vector spaces into direct sums of subspaces. (The decomposition needs not be orthogonal. In fact, an inner product may not even be defined.)

(2) An appropriate choice of basis vectors: Any subspace must be spanned by a subset of the basis vectors.

## 8. Isometric mappings between inner product spaces.

### 8.1. Definitions.

Let V and W be vector spaces and let $\Lambda$ be a linear operator from V into W:

$$y = \Lambda(x),\ x\epsilon V,\ y\epsilon W$$

The mapping is called isometric if

$$\|\Lambda(x)\| = \|x\|$$

It follows that no vector $x\neq 0$ can be mapped onto the zero vector. If $\Lambda$ maps V onto W, the mapping is invertible.

### 8.2. Preservation of inner products.

Isometric mappings preserve the inner product. For let

$$y_1 = \Lambda(x_1),\quad y_2 = \Lambda(x_2)$$

Then $\|y_1-y_2\| = \|x_1-x_2\|$ implies

$$(y_1-y_2,y_1-y_2) = (x_1-x_2,x_1-x_2)$$

or

$$(y_1,y_1) - 2(y_1,y_2) + (y_2,y_2) = (x_1,x_1) - 2(x_1,x_2) + (x_2,x_2)$$

Due to isometry we have $(y_1,y_1) = (x_1,x_1)$, $(y_2,y_2) = (x_2,x_2)$. Hence

$$(y_1, y_2) = (x_1, x_2)$$

8.3 Matrix representation.

Let V and W be of dimension n. Choose bases in V and W. Let the positive definite matrices $G_V$ and $G_W$ represent the inner products in V,W. Let the nxn matrix A represent the operator $\Lambda$. Identify vectors with their coordinate n-tuples. We write as usual

$$(x_1, x_2) = x_1^T G_V x_2, \qquad x_1, x_2 \in V$$
$$(y_1, y_2) = y_1^T G_W y_2, \qquad y_1, y_2 \in W$$

The isometry requirement implies for any $x_1, x_2 \in V$:

$$x_1^T G_V x_2 = (Ax_1)^T G_W (Ax_2) = x_1^T A^T G_W A x_2$$

It follows that

$$G_V = A^T G_W A$$

8.4. Examples of isometric mappings.

8.4.1. The isometric mapping between any vector space V and its dual V', the space of linear functionals. We refer to section 2.6. The transformation matrix equals $G_V$. For if the vector x is mapped onto the functional $\xi$, we have $\xi = G_V x$. The matrix $G_{V'}$ equals $K_V = G_V^{-1}$.

8.4.2. Change of basis in V. Let $e_1,\ldots,e_n$ be the old basis in V, and G represent the inner product. Let $e'_1,\ldots,e'_n$ be the new basis, and G' be the corresponding representation of the inner product in V. As shown in section 3.2, the relation between old coordinate vectors x and new ones x' is

$$x = Ax'$$

(The j-th column of A contains the scalar factors expressing $e'_j$ in terms of $e_i$, i=1,...,n.).

Since the inner product is a property of V and not of any basis, it must be preserved during transformation

$$x^TGy = (Ax')^TG(Ay') = x'^T(A^TGA)y' = x'^TG'y'$$

It follows that

$$G' = A^TGA$$

8.4.3. Isometry between a subspace and the space of its parameters. Let $V_A$ be an m-dimensional subspace of the n-dimensional space $V_n$. Let $V_A$ be spanned by the linearly independent columns of the nxm matrix A. Let the matrix G represent the inner product in $V_n$. $V_A$ has an inner product inherited from $V_n$:

$$(a_1,a_2) = a_1^TGa_2$$

Any vector $a \in V_A$ is uniquely represented by its coordinates with respect to the columns of A. These columns may be viewed as a basis of $V_A$. Thus the system of equations

$$a = Ax$$

has a unique solution x. There is a one to one mapping between $V_A$ and the space X of m-dimensional coordinate vectors x. In order to preserve the inner product, one must require

$$(x_1, x_2) = (a_1, a_2), \quad a_1 = Ax_1, \; a_2 = Ax_2$$

Letting $G_X$ denote the matrix of the inner product in X, one finds

$$(x_1, x_2) = x_1^T G_X x_2^T = (a_1, a_2) = x_1^T A^T G A x_2$$

Thus

$$G_X = A^T G A$$

This looks the same as before; however this time A is not invertible.

8.5. Canonical transformation of an adjustment problem.

We consider adjustment by variation of parameters:

$$\ell + v = Ax \quad \text{with weight matrix } P$$

We denote by L the space of observations and by $L_A$ that of the adjusted observations. We choose an orthonormal basis in $L_A$. Let the matrix $\bar{A}$ comprise the new orthonormal vectors. Confer section 4.7. The columns of $\bar{A}$ are expressible in terms of the columns of A:

$$\bar{A} = AC$$

We introduce new parameters

$$x = Cy$$

The new adjustment problem is

$$\ell+v = ACy = \bar{A}y$$

Consider the matrix $\bar{B}$ having orthonormal columns and spanning the orthocomplement of $V_A$. We have

$$\bar{A}^T P \bar{A} = I,\ \bar{B}^T P \bar{B} = I,\ \bar{A}^T P \bar{B} = 0$$

Consider an isometric transformation to new observations $\ell'$ given by

$$\begin{bmatrix} \ell_1' \\ \ell_2' \end{bmatrix} = \begin{bmatrix} \bar{A}^T P \\ \bar{B}^T P \end{bmatrix} \ell$$

The matrices

$$\begin{bmatrix} \bar{A}^T P \\ \bar{B}^T P \end{bmatrix}$$

and

$$(\bar{A}, \bar{B})$$

are inverse to each other. This is easily seen by multiplying these two partitioned matrices and minding the orthogonality relations between $\bar{A}$ and $\bar{B}$.

Premultiply the adjustment problem by

$$\begin{bmatrix} \bar{A}^T P \\ \bar{B}^T P \end{bmatrix}$$

To obtain

$$\ell_1' + v_1' = Iy$$
$$\ell_2' + v_2' = 0$$

The weight matrix of the new observations $\ell'$ is obtained as

$$P' = \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} P \quad (\bar{A}, \bar{B}) = I$$

The solution of the canonically transformed adjustment problem is obviously

$$\ell_1' = y, \quad v_1' = 0$$

$$\ell_2' = 0, \quad v_2' = -\ell_2'$$

Remark: Deriving the canonical form requires no less computational work than solving the adjustment problem conventionally. The benefit is theoretical insight.

## 9. Partial reduction.

### 9.1 Partitioning the set of parameters.

Consider an adjustment problem by variation of parameters

$$\ell+v = Ax, \qquad \text{weight matrix } P$$

Assume A being an n×m matrix. The m-dimensional vector of parameters x is partitioned into an $m_1$-dimensional vector $x_1$ and an $m_2$-dimensional vector $x_2$:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Of course, $m_1 + m_2 = m$. Partition the columns of A accordingly

$$A = (A_1, A_2)$$

The adjustment problem is then written as

$$\ell+v = (A_1, A_2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\ell+v = A_1x_1 + A_2x_2$$

We are primarily interested in adjusted values of $x_2$. The parameters $x_1$ play an auxiliarly role, as for example orientation unknowns.

## 9.2 Partial reduction of the normal equations.

The normal equations are

$$(A^T PA)x = A^T P\ell$$

or briefly

$$Gx = r$$

The partitioning of A induces a partitioning of G and r

$$\begin{bmatrix} A_1^T PA_1 & A_1^T PA_2 \\ A_2^T PA_1 & A_2^T PA_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1^T P\ell \\ A_2^T P\ell \end{bmatrix}$$

We abbreviate this as

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

This may also be written

$$G_{11}x_1 + G_{12}x_2 = r_1$$
$$G_{21}x_1 + G_{22}x_2 = r_2$$

We may eliminate $x_1$ from these equations in the same way as this was done

in section 7.4. We obtain the partially reduced set of normal equations for $x_2$:

$$(G_{22}-G_{21}G_{11}^{-1}G_{12})x_2 = r_2 - G_{21}G_{11}^{-1}r_1$$

This is abbreviated as

$$\bar{G}_{22}x_2 = \bar{r}_2$$

Our intention is to understand these equations geometrically.

## 9.3. Orthocomplementary decomposition of the space of adjusted observations and its parameter space.

Let L denote the space of observations, and let $L_A$ be the space of adjusted observations. $L_A$ is spanned by the columns of A. We now decompose $L_A$ into $L_{A_1}$, the space spanned by the columns of $A_1$, and into $L_{\bar{A}_2}$. $L_{\bar{A}_2}$ is the orthocomplement of $L_{A_1}$ in $L_A$. It is spanned by the columns of a matrix $\bar{A}_2$ which is yet to be determined. The following relations must hold

$$(A_1, \bar{A}_2) \text{ span } L_A$$
$$A_1^T P \bar{A}_2 = 0$$

We use the isometry between $L_A$ and its parameter space X. Confer section 8.4.3. The columns of A are mapped onto the natural basis of X. The inner product in X is represented by

$$G = A^T P A$$

Any vector x in X is represented as

$$x = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ I \end{bmatrix} x_2$$

i.e.

$$x = J_1 x_1 + J_2 x_2$$

X is the direct sum of two subspaces $X_1$ and $X_2$ spanned by the columns of $J_1$, $J_2$. These subspaces are not orthogonal. We have

$$J_i^T G J_j = G_{ij}, \quad i,j=1,2$$

We replace $J_2$ by $\bar{J}_2$ orthogonal to $J_1$. We proceed formally in a similar fashion as in section 4.8 on Gram-Schmid orthogonalization. Just confer the way, the second vector $\bar{a}_2$ was derived there. We represent:

$$\bar{J}_2 = J_2 - J_1 C$$

where the matrix C is yet to be determined. Requiring $J_1^T G \bar{J}_2 = 0$ leads to

$$0 = G_{12} - G_{11} C, \quad \text{i.e.} \quad C = G_{11}^{-1} G_{12}$$

Thus

$$\begin{aligned} \bar{J}_2 &= J_2 - J_1 G_{11}^{-1} G_{12} = \\ &= J_2 - J_1 (J_1^T G J_1)^{-1} J_1^T G J_2 \\ &= J_2 - P_{J_1} J_2 = (I - P_{J_1}) J_2 \end{aligned}$$

We view the columns of $(J_1, \bar{J}_2)$ as a new basis in X. A vector x is represented as

$$x = J_1 y_1 + \bar{J}_2 y_2$$

Inserting for $\bar{J}_2$ we get

$$x = (J_1 y_1 - P_{J_1} J_2 y_2) + J_2 y_2$$

The vector in parentheses is in $X_1$, (spanned by $J_1$), the second vector on the right hand side is in $X_2$ (spanned by $J_2$). A comparision with the earlier representation $x = J_1 x_1 + J_2 x_2$ shows:

$$J_1 x_1 = J_1 y_1 - J_1 G_{11}^{-1} G_{12} y_2$$
$$J_2 x_2 = J_2 y_2$$

Because the columns of $J_1$ and $J_2$ are linearly independent, we get

$$x_1 = y_1 - G_{11}^{-1} G_{12} y_2$$
$$x_2 = y_2$$

This expresses the old coordinates of the vector x in terms of the new ones. The orthocomplementary decomposition of X into spaces spanned by $J_1$, $\bar{J}_2$ induces an orthocomplementary decomposition of $L_A$ into spaces spanned by $A_1$, $\bar{A}_2$. A vector $a \in L_A$ was previously represented as

$$a = A_1 x_1 + A_2 x_2$$

now it is represented by

$$a = A_1 y_1 + \bar{A}_2 y_2$$

The new representation is obtained either by substituting for $x_1$, $x_2$:

$$\begin{aligned} a &= A_1(y_1 - G_{11}^{-1} G_{12} y_2) + A_2 y_2 \\ &= A_1 y_1 + (A_2 - A_1 G_{11}^{-1} G_{12}) y_2 \\ &= A_1 y_1 + (A_2 - A_1 (A_1^T P A_1)^{-1} A_1^T P A_2) y_2 \\ &= A_1 y_1 + (I - P_{A_1}) A_2 y_2 \\ &= A_1 y_1 + \bar{A}_2 y_2 \end{aligned}$$

or by noting that the dependence of $\bar{A}_2$ on $A_1$, $A_2$ must be the same as that of $\bar{J}_2$ on $J_1$, $J_2$:

$$\bar{A}_2 = A_2 - A_1 G_{11}^{-1} G_{12} = A_2 - A_1 (A_1^T P A_1)^{-1} A_1^T P A_2 = (I - P_{A_1}) A_2$$

In any case, the desired matrix $\bar{A}_2$ is obtained as

$$\bar{A}_2 = (I - P_{A_1}) A_2$$

Our adjustment problem is thus transformed into

$$\ell+v = A_1 y_1 + \bar{A}_2 y_2 = A_1 y_1 + \bar{A}_2 x_2$$

because $y_2 = x_2$.

Verify that $\bar{A}_2^T P \bar{A}_2 = G_{22} - G_{21} G_{11}^{-1} G_{12} = \bar{G}_{22}$, $\bar{A}_2^T P \ell = r_2 - G_{21} G_{11}^{-1} r_1 = \bar{r}_2$. Hence the transformed normals are found to be:

$$G_{11} y_1 = r_1$$
$$\bar{G}_{22} x_2 = \bar{r}_2$$

they decompose into two independent sets. The second one is identical to the partially reduced normals for $x_2$.

The partially reduced normals give $x_2$. The question remains how to find the residuals v without calculating $y_1$ from the first set of the above equations.

9.4. The partially reduced observation equations.

If we did solve the complete set of transformed normals, we would get v from

$$\ell+v = A_1 y_1 + \bar{A}_2 x_2$$

Here

$$y_1 = G_{11}^{-1} r_1 = (A_1^T P A_1)^{-1} A_1^T P \ell$$

We see that

$$\ell - A_1(A_1^T P A_1)^{-1} A_1^T P \ell + v = \bar{A}_2 x_2$$

or

$$\ell - P_{A_1}\ell + v = \bar{A}_2 x_2$$
$$(I - P_{A_1})\ell + v = \bar{A}_2 x_2$$
$$\bar{\ell} + v = \bar{A}_2 x_2$$

The last set is called partially reduced observation equations. They involve the pseudo observations

$$\bar{\ell} = (I - P_{A_1})\ell$$

It is important to note that the normals obtained from the partially reduced observation equations are just the partially reduced normals for $x_2$.

$$(\bar{A}_2^T P \bar{A}_2)x_2 = \bar{A}_2^T P \bar{\ell} = \bar{A}_2^T P(\ell - P_{A_1}\ell) = \bar{A}_2^T P \ell \quad \text{i.e. } \bar{G}_{22} x_2 = \bar{r}_2$$

(Mind that $\bar{A}_2$ is orthogonal to $P_{A_1}\ell$)

9.5. Alternative derivation of the partially reduced observation equations.

Consider the orthocomplementary decomposition of L into 3 subspaces $L_{A_1}$, $L_{\bar{A}_2}$, $L_B$. The spaces $L_{A_1}$, $L_{\bar{A}_2}$ are already familiar. They are spanned by the columns of the matrices $A_1$, $\bar{A}_2$. The space $L_B$ is the orthocomplement of the direct sum of these spaces. It is also the orthocomplement of $L_A$, $A = (A_1, A_2)$. The space $L_B$ is spanned by the columns of the matrix B. It holds that

$$A_1^T P \bar{A}_2 = 0, \quad A_1^T P B = 0, \quad \bar{A}_2^T P B = 0$$

Let a new basis of L be given by the union of the columns of $A_1$, $\bar{A}_2$, B. We transform the observations $\ell$ to the new basis:

$$\ell = (A_1, \bar{A}_2, B) \begin{bmatrix} \ell_1' \\ \ell_2' \\ \ell_3' \end{bmatrix}$$

Because the subspaces $L_{A_1}$, $L_{\bar{A}_2}$, $L_B$ are orthogonal, it also holds that

$$A_1\ell_1 = P_{A_1}\ell, \quad \bar{A}_2\ell_2 = P_{\bar{A}_2}\ell, \quad B\ell_3 = P_B\ell$$

Inserting the matrix representations of the projectors (e.g. $P_{A_1} = A_1(A_1^T P A_1)^{-1} A_1^T P$ ) one finds the formula expressing $\ell'$ in terms of $\ell$:

$$\begin{bmatrix} \ell_1' \\ \ell_2' \\ \ell_3' \end{bmatrix} = \begin{bmatrix} (A_1^T P A_1)^{-1} A_1^T P \\ (\bar{A}_2^T P \bar{A}_2)^{-1} \bar{A}_2^T P \\ (B^T P B)^{-1} B^T P \end{bmatrix} \ell$$

The weight matrix of the new observations is the representation of P with respect to the new basis. One finds

$$\begin{bmatrix} A_1^T P A_1 & 0 & 0 \\ 0 & \bar{A}_2^T P \bar{A}_2 & 0 \\ 0 & 0 & B^T P B \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & G_{22} & 0 \\ 0 & 0 & B^T P B \end{bmatrix}$$

It is important to note the blockdiagonal structure. The observation equations transform as:

$$\ell_1' + v_1' = y_1$$
$$\ell_2' + v_2' = x_2$$
$$\ell_3' + v_3' = 0$$

The 3 subproblems for $\ell_1'$, $\ell_2'$, $\ell_3'$ are completely independent. Any subproblem has separate observations, corrections and unknowns. Essential is also the block diagonal structure of the transformed weight matrix. There is no coupling of the subproblems due to weights. One immediately obtains the solution

$$y_1 = \ell_1', \quad v_1' = 0$$
$$x_2 = \ell_2', \quad v_2' = 0$$
$$v_3' = -\ell_3'$$

Because the 3 subproblems are independent, the result for $x_2$ is unaffected if we put

$$\ell_1' = 0, \quad y_1 = 0$$

Hence the problem

$$0 + v_1' = 0$$
$$\ell_2' + v_2' = x_2$$
$$\ell_3' + v_3' = 0$$

yields correct results for $x_2$ and $v_1'$, $v_2'$, $v_3'$. Transforming this problem backward to the old basis, one obtains

$$A_1 \ell_1' + B\ell_3' + v = \bar{A}_2 x_2$$

or

$$P_{A_1} \ell + P_B \ell + v = \bar{A}_2 x_2$$

or

$$(I - P_{\bar{A}_2})\ell + v = \bar{A}_2 x_2$$

These are the partially reduced observation equations.

## 10. Adjustment phased with respect to observations.

### 10.1 Formulation of the problem.

Consider an adjustment problem by variation of parameters.

$$\ell + v = Ax, \quad \text{weight matrix } P$$

Decompose the vector $\ell$ into two subvectors. Decompose v and A, accordingly

$$\ell = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Assume that the weight matrix P has the following special structure

$$P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$$

There are no weights coupling the two groups of observations.

One may imagine that the two groups of observations refer to different time periods. It could also be that they refer to different geographical regions with partial overlap. In any case, one can consider the two separate adjustment problems

$$\ell_1 + v_1 = A_1 x \quad \text{weight matrix } P_{11}$$
$$\ell_2 + v_2 = A_2 x \quad \text{weight matrix } P_{22}$$

and one can obtain their separate solutions $x_{(1)}$, $x_{(2)}$. The question is, how the solution of the entire problem is related to these partial solutions.

One can also proceed differently. Suppose that the observations $\ell_1$ are available earlier. Then one calculates $x_{(1)}$ from the first set of the above relations. This is phase 1 of the adjustment. Subsequently observations $\ell_2$ become available. One is then interested to calculate in the second phase the solution x of the entire problem by using $x_{(1)}$, the solution of the previous phase in combination with the observations $\ell_2$ of the second phase. Of course, one can consider more than two phases. However the essential features of phased adjustment become transparent if only two phases are considered.

## 10.2 Addition of normal equations.

The normal equations of the two separate problems are

$$(A_1^T P_{11} A_1)x = A_1^T P_{11} \ell_1$$
$$(A_2^T P_{22} A_2)x = A_2^T P_{22} \ell_2$$

The normal equations of the entire problem are

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$$

This is evaluated as

$$(A_1^T P_{11} A_1 + A_2^T P_{22} A_2)x = (A_1^T P_{11} \ell_1 + A_2^T P_{22} \ell_2)$$

It is seen that the normals of the entire problem are obtained by adding the normals of the two phases. The added normals can be solved to give the solution x.

10.3 Updating the solution of the previous phase.

Consider the problem of the first phase

$$\ell_1 + v_1 = A_1 x$$

Its solution is

$$x_{(1)} = (A_1^T P_{11} A_1)^{-1} A_1^T P_{11} \ell_1$$

Let $L_1$ be the space of observations $\ell_1$. We consider the subspace $L_{A_1}$ of $L_1$ spanned by the columns of $A_1$. We decompose $L_1$ into orthocomplementary subspaces $L_{A_1}$ and $L_{B_1}$. The space $L_{B_1}$ is spanned by columns of $B_1$. The residuals of phase 1 are

$$v_{(1)} = A_1 x_{(1)} - \ell_1$$

As shown in section 6.1, $v_{(1)}$ may be represented as

$$v_{(1)} = -B(B_1^T P_{11} B_1)^{-1} B_1^T P_{11} \ell_1$$

The second representation for $v_{(1)}$ will only be needed for the purpose of mathematical proofs.

Because $L_{B_1}$ is the orthocomplement of $L_{A_1}$, we have

$$A_1^T P_{11} B_1 = 0$$

Consider now a transformation of the observations of phase 1.

$$\ell'_{11} = (A_1^T P_{11} A_1)^{-1} A_1^T P_{11} \ell_1 = x_{(1)}$$
$$\ell'_{12} = (B_1^T P_{11} B_1)^{-1} B_1^T P_{11} \ell_2$$

Note that the two matrices

$$\begin{bmatrix} (A_1^T P_{11} A_1)^{-1} A_1^T P_{11} \\ (B_1^T P_{11} B_1)^{-1} B_1^T P_{11} \end{bmatrix} \quad \text{and } (A_1, B_1)$$

are inverse to each other. (Multiply them to obtain the unit matrix I). Hence the transformation is equivalent to

$$\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = (A_1, B_1) \begin{bmatrix} \ell'_{11} \\ \ell'_{12} \end{bmatrix}$$

The weight matrix of the transformed observations is given by

$$P' = (A_1, B_1)^T P (A_1, B_1) = \begin{bmatrix} A_1^T P A_1 & 0 \\ 0 & B_1^T P B_1 \end{bmatrix}$$

The transformed adjustment problem is

$$\ell'_{11} + v'_{11} = x$$
$$\ell'_{12} + v'_{12} = 0$$

Equivalently

$$x_{(1)} + v'_{11} = x, \quad \text{weight matrix} \quad A_1^T P_{11} A_1$$
$$\ell'_{12} + v'_{12} = 0, \quad \text{weight matrix} \quad B_1^T P_{11} B_1$$

The two problems are independent, because the weight matrix is block diagonal. All the information on x available from phase 1 is contained in the first set

$$x_{(1)} + v'_{11} = x \quad , \quad \text{weight matrix} \quad A_1^T P_{11} A_1$$

We add the observation equations of the second phase, arriving at the problem

$$x_{(1)} + v'_{11} = x$$
$$\ell_2 + v_2 = A_2 x$$

with weight matrix

$$\begin{bmatrix} A_1^T P_{11} A_1 & 0 \\ 0 & P_{22} \end{bmatrix}$$

Forming the normals gives precisely the normals of the entire problem as they were obtained earlier:

$$(A_1^T P_{11} A_1 + A_2^T P_{22} A_2)x = (A_1^T P_{11} \ell_1 + A_2^T P_{22} \ell_2)$$

(The calculation of the normals is easy and is omitted.) The solution x is calculated. One calculates residuals of the second phase.

$$v_{11}' = x - x_{(1)}$$
$$v_2 = A_2 x - \ell_2$$

The following relationship between residuals $v_{(1)}$, $v_{11}'$ and the residuals

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

of the original problem is interesting.

Theorem:

$$v^T P v = v_1^T P_{11} v_1 + v_2^T P_{22} v_2 =$$
$$= v_{(1)}^T P_{11} v_{(1)} + v_{11}'^T A_1^T P_{11} A_1 v_{11}' + v_2^T P_{22} v_2$$

Equivalently

$$v^T P v = (\ell_1 - Ax_{(1)})^T P_{11} (\ell_1 - Ax_{(1)}) +$$
$$+ (x - x_{(1)})^T A_1^T P_{11} A_1 (x - x_{(1)}) +$$
$$+ (\ell_2 - A_2 x)^T P_{22} (\ell_2 - A_2 x)$$

The first term on the right hand side is the weighted sum of residuals obtained from phase 1. The second and third term comprise the weighted sum of residuals from phase 2. We thus obtain:

"The weighted sum of residuals of the combined phases is the sum of the weighted sums of residuals of the individual phases."

Proof: The theorem of Pythagoras applied to the entire problem and to the individual phases gives the following three relations. (Confer section 6.3.)

$$v_1^T P_{11} v_1 + v_2^T P_{22} v_2 = \ell_1^T P_{11} \ell_1 + \ell_2^T P_{22} \ell_2 - x^T A_1^T P_{11} A_1 x - x^T A_2^T P_{22} A_2 x$$
$$v_1^T P_{11} v_1 = \ell_1^T P_{11} \ell_1 - x_{(1)}^T A_1^T P_{11} A_1 x_{(1)}$$
$$v_{11}^T A_1^T P_{11} A_1 v_{11} + v_2^T P_{22} v_2 = x_{(1)}^T A_1^T P_{11} A_1 x_{(1)} +$$
$$+ \ell_2^T P_{22} \ell_2 - x^T A_1^T P_{11} A_1 x - x^T A_2^T P_{22} A_2 x$$

It is seen that the first relation is the sum of the second and third. This proves the theorem.

10.4 Geometrical insight.

The space L of observations is represented as the direct sum of two orthocomplementary subspaces $L_1$, $L_2$. Bases in $L_1$ and $L_2$ are chosen and

subsequently combined to a basis of L. This allows us to use the calculus of partitioned matrices as outlined in section 7.5. The matrix of the inner product necessarily decomposes into blockdiagonal form.

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

because any vector in $L_1$ is orthogonal to any vector in $L_2$.

We consider the space $L_A$, spanned by the columns of $A_1$. It is a subspace of $L_1$. The subspace $L_1$ is viewed as the direct sum of $L_{A_1}$ and its orthocomplement $L_{B_1}$ in $L_1$. A change of basis in $L_1$ is performed such that the columns of $A_1$ and $B_1$ become basis vectors. This induces the transformation $\ell_1 \rightarrow \ell_1'$. The entire space of observations L is now the direct sum of three orthocomplementary subspaces $L_{A_1}$, $L_{B_1}$, $L_2$. The space of adjusted observations $L_A$ is only participating in $L_{A1}$, $L_2$. Only the zero vector is common to $L_A$ and $L_{B_1}$. Hence the space $L_{B_1}$ may be ignored in the adjustment problem. This leads to the simplified setup of the second phase.

$$\begin{aligned} x_{(1)} + v_{11} &= x \\ \ell_2 + v_2 &= A_2 x \end{aligned}$$

with weight matrix

$$\begin{bmatrix} A_1^T P A_1 & 0 \\ 0 & P_{22} \end{bmatrix}$$

10.5 Pre-elimination of group-internal unknowns.

Phased adjustment as outlined above is not very effective from the viewpoint of computational efficiency. It becomes a powerful tool if it is combined with partial reduction as presented in chapter 9. A great benefit arises if there are sets of auxiliary unknowns, each one referring to only one group of observations. If auxiliary unknowns $y_i$ are only present in group i, we call them "group-internal" unknowns. It suffices to consider the setup

$$\begin{aligned} \ell_1 + v_1 &= H_1 h_1 + A_1 x \\ \ell_2 + v_2 &= H_2 h_2 + A_2 x \end{aligned} \qquad \text{weight matrix } P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$$

Here $h_1$ are auxiliary unknowns which are internal to group 1. The unknowns $h_2$ are internal to group 2. $H_1$ and $H_2$ are the corresponding design matrices.

A remarkable simplification of the computation results if the group internal unknowns are eliminated before the groups are combined.

As outlined in chapter 9., the elimination can be accomplished in two different ways. If

$$\ell_i + v_i = H_i h_i + A_i x$$

are the observation equations, one may either form the partially reduced observation equations

$$\bar{\ell}_i + v_i = \bar{A}_i x, \quad \bar{A}_i = A_i - P_{H_i} A_i$$

which lead to the partially reduced normals

$$\bar{A}_i^T P_{ii} \bar{A}_i x = \bar{A}_i^T P_{ii} \ell$$

or one may form the normals for a group

$$\begin{bmatrix} H_i^T P_{ii} H_i & H_i^T P_{ii} A_i \\ A_i^T P_{ii} H_i & A_i^T P_{ii} A_i \end{bmatrix} \begin{bmatrix} h_i \\ x \end{bmatrix} = \begin{bmatrix} H_i^T P \ell \\ A_i^T P \ell \end{bmatrix}$$

and eliminate the auxiliary unknowns $h_i$. The result will be the same set of partially reduced normals, although the immediately derived expression looks differently, namely as

$$(A_i^T P_{ii} A_i - A_i^T P_{ii} H_i (H_i^T P_{ii} H_i)^{-1} H_i^T P_{ii} A_i) x =$$
$$A_i^T P \ell - A_i^T P_{ii} H_i (H_i^T P_{ii} H_i)^{-1} H_i^T P \ell, \qquad i=1,2$$

In any case, the partially reduced normals

$$\bar{G}_{ii} x = \bar{r}_i$$

of all groups may be added to give the partially reduced normals of the entire system

$$(\bar{G}_{11} + \bar{G}_{22}) x = \bar{r}_1 + \bar{r}_2$$

A proof for the validity of the procedure of group-wise elimination of group-internal unknowns can certainly be given in geometrical terms. Occasionally, however, it is preferable to use calculus. We have to show that the same result is obtained if partial reduction is done in the conventional way, i.e. by ignoring the block decomposition of our system resulting from decomposing $\ell$ into $\ell_1$ and $\ell_2$. Write the observation equations as

$$\ell + v = Hh + Ax, \quad \text{weight matrix } P$$

whereby

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$$

In agreement with section 9.5 we obtain the partially reduced observation equations as

$$\bar{\ell} + v = \bar{A}x, \quad \text{weight matrix } P$$

with

$$\bar{\ell} = (I-P_H)\ell, \quad \bar{A} = (I-P_H)A$$

Forming $P_H$, one recognizes

$$P_H = \begin{bmatrix} P_{H_1} & 0 \\ 0 & P_{H_2} \end{bmatrix}, \qquad P_{H_i} = H_i(H_i^T P_{ii} H_i)^{-1} H_i^T P_{ii}, \qquad i=1,2$$

Hence

$$\bar{\ell} = \begin{bmatrix} \bar{\ell}_1 \\ \bar{\ell}_2 \end{bmatrix}, \quad \text{with} \quad \bar{\ell}_i = (I-P_{H_i})\ell_i,$$

$$\bar{A} = \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \end{bmatrix}, \quad \text{with} \quad \bar{A}_i = (I-P_{H_i})A_i$$

These are precisely the quantities occurring in the partially reduced observation equations obtained by considering the two groups separately.

This completes the proof for the case of group-wise partially reduced observation equations. The proof for the validity of the group-wise partially reduced normals is even simpler. It is omitted.

Remark: A geometric proof would start from the afore mentioned decomposition of the vector space L into a direct sum of orthocomplementary subspaces $L_1$ and $L_2$. In analogy to section 9.6 each subspace $L_i$, i=1,2, is further decomposed into 3 orthocomplementary spaces $L_{H_i}$, $L_{\bar{A}_i}$ and $L_{B_i}$. One considers $L_H$ as the direct sum of $L_{H_1}$ and $L_{H_2}$. The projector $\Pi_H$ onto $L_H$ is represented as

$$\Pi_H = \Pi_{H_1} + \Pi_{H_2}$$

Because $L_{H_i} \subset L_i$, i=1,2, this may also be written

$$\Pi_H = \Pi_{L_1} \circ \Pi_{H_1} \circ \Pi_{L_1} + \Pi_{L_2} \circ \Pi_{H_2} \circ \Pi_{L_2}$$

Using this together with

$$\bar{\ell} = (I-\Pi_H)\ell, \quad \ell_i = \Pi_{L_i}\ell, \quad \bar{\ell}_i = \Pi_{L_i}\bar{\ell}$$

one gets

$$\bar{\ell}_i = \Pi_{L_i} \circ (I-\Pi_{H_i})\ell_i$$

In agreement with section 7.5, one recognizes that the operators $\Pi_{L_i} \circ (I-\Pi_{H_i})$ map $L_i$ into $L_i$, i=1,2. They map $L_j$, j≠i, onto zero. With respect to the basis of $L_i$, those operators are represented by the matrices

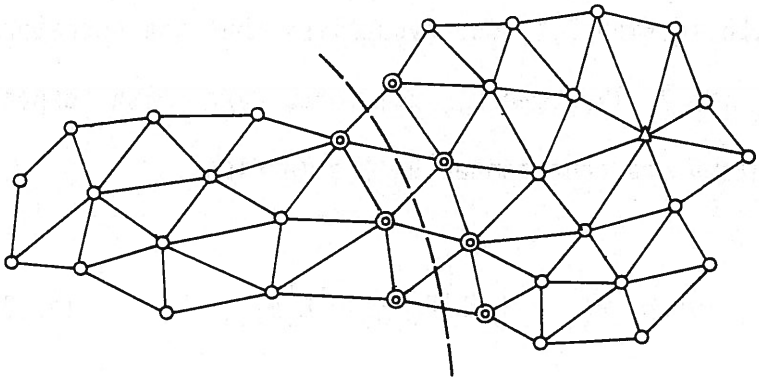
$$I-\Pi_{H_i}, \quad \text{with} \quad \Pi_{H_i} = H_i(H_i^T P_{ii} H_i)^{-1} H_i^T P_{ii}, \qquad i=1,2$$

encountered earlier.

10.6 Helmert blocking.

The procedure of section 10.5 is the theoretical basis for Helmert blocking. As an example take a network as depicted in the subsequent figure. Let $\ell_1$ denote observations taken at stations to the east of the dashed line. Let $\ell_2$ denote the observations taken at stations to the right. Let $h_1$ comprise coordinate increments of stations marked by single circles and situated to the east of the dividing line. Such stations are called inner stations of the eastern block. Include in $h_1$ also orientation unknowns of observations out of $\ell_1$. Define $h_2$

accordingly. Finally let x denote coordinate increments of junction stations marked by double circles. The procedure of section 10.5 gives rigorously adjusted values of the junction station coordinate increments.



## 11. Complementary extremum principles in least squares adjustment.

### 11.1 The basic geometric principle.

Let V be an n-dimensional vector space equipped with an inner product. Let $V_A$, $V_B$ be orthocomplementary subspaces. The following theorem is a near triviality.

Theorem: Let $a \in V_A$. The vector $b \in V_B$ closest to a is the zero vector b=0.

Proof: $d(a,b)^2 = \|a-b\|^2 = (a-b,a-b) = (a,a)+(b,b)$, because $(a,b) = 0$, for $a \in V_A$, $b \in V_B$. Thus $d(a,b) = \|a\|^2 + \|b\|^2$. Obviously this is minimal for b=0.

### 11.2 Reformulation for linear manifolds.

We shift the problem slightly away from triviality by considering linear manifolds instead of subspaces.

Definition: Let $V_A$ be a subspace and let $a_0$ be any fixed vector in V. The linear manifold $M_A$ comprises all vectors u which may be represented as

$$u = a_0 + a, \quad a \in V_A$$

Similarly, the linear manifold $M_B$ is introduced. It comprises all vectors representable as

$$v = b_0 + b, \quad b \in V_B$$

$b_0$ is again a fixed vector in V. It is seen that a linear manifold is generally not a subspace. The zero vector may not be a member of $M_A$ or $M_B$. However,

difference vectors of the vectors in a linear manifold form a vector-subspace.

Next we show that the linear manifolds $M_A$ and $M_B$ have only one vector in common if the participating subspaces are orthocomplementary. Let

$$w = a_0 + a = b_0 + b, \quad a \in V_A, \; b \in V_B$$

It follows that

$$b_0 - a_0 = a - b, \quad a \in V_A, \; b \in V_B$$

The decomposition of $b_0 - a_0$ into vectors of $V_A$ and $V_B$ is unique. This shows existence and uniqueness of w. It also shows that

$$a = \Pi_A(b_0 - a_0)$$
$$b = \Pi_B(a_0 - b_0)$$

Here $\Pi_A$ and $\Pi_B$ are the (orthogonal) projection operators onto the subspaces $V_A$, $V_B$.

The translation

$$x' = x + w$$

carries the linear subspaces $V_A$, $V_B$ over into the manifolds $M_A$, $M_B$. Because distances are translation-invariant, the theorem of section 11.1 is reformulated

as follows.

Theorem: Let $u \in M_A$. Then the vector $v \in M_B$ closest to u is the vector w, representing the intersection of $M_A$ and $M_B$.

The roles of $M_A$ and $M_B$ may be interchanged. Thus it is seen that w is the solution of two extremum problems:

(I) Given $v \in M_B$, find $u \in M_A$ such that

$$d(u,v)^2 = \text{Minimum}$$

(II) Given $u \in M_A$, find $v \in M_B$ such that

$$d(u,v)^2 = \text{Minimum}$$

The two extremum problems have different admissible sets, namely $M_A$ and $M_B$. The only vector common to both admissible sets solves both extremum problems.

Suppose that u' is admissible for (I), but not necessarily optimal. Similarly let v' be admissible for (II). Then

$$d(v,w)^2 \leq d(v,u')^2$$
$$d(u,w)^2 \leq d(u,v')^2$$

By the theorem by Pythagoras

$$d(u,v)^2 = d(u,w)^2 + d(v,w)^2$$

Using the second of the above inequalities, one finds

$$d(v,w)^2 \geq d(u,v)^2 - d(u,v')^2$$

Combining with the first of the above inequalities one gets a lower and an upper bound on the optimum value of (I):

$$d(u,v)^2 - d(u,v')^2 \leq d(v,w)^2 \leq d(v,u')^2$$

Similarly, an inclusion of the extremum of (II) is obtained. This is also obvious if one notes that both extrema sum up to $d(u,v)^2$.

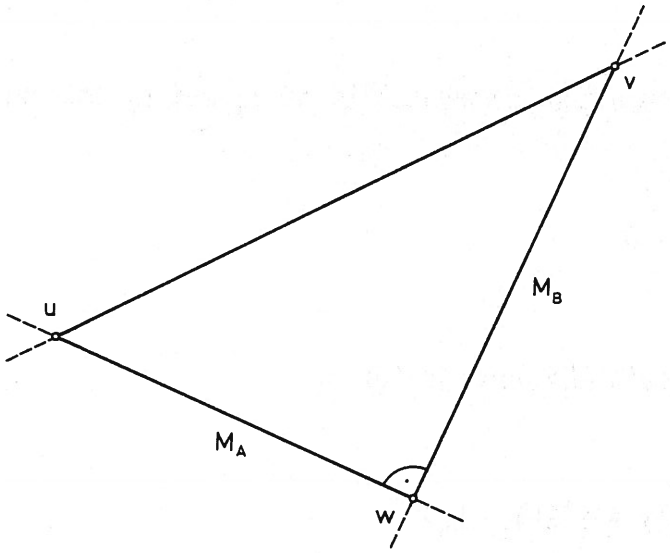
Remark: In the literature, the second problem is frequently posed as follows:
(II*) Given $u \in M_A$ and $v \in M_B$, find $w \in M_B$ such that

$$d(u,v)^2 - d(u,w)^2 = \text{Maximum}$$

This redefinition causes the two optima to coincide. ("The energy equals the complementary energy"). Besides this, the second problem now searches for a maximum. The solution is therefore a minimum for (I) and a maximum for (II*). It is a saddle point. It is also more obvious now, why admissible vectors for (I)

and (II*) give upper and lower bounds on the optimum. On the other hand, a certain lack of symmetry in (I) and (II*) is noted. Therefore we prefer our original setup.

Remark: In 2 dimensions the complementary extremum principles and their solutions can be read off from a rectangular triangle




The points u, v form the end points of the hypotenuse. Problem (I) searches for a point on the straight line containing the short side from u to w. The point shall be nearest to v. Obviously the solution is w. Problem (II) is obtained by symmetry.

Remark: If the subspace $V_A$ is spanned by the linearly independent columns of the matrix A, and if $V_B$ is spanned similarly by the columns of B, then w is represented as

$$w = a_0 + Ax = b_0 + By$$

we get

$$Ax - By = b_0 - a_0$$

Premultiplying by $A^TG$, where G represents the inner product, we obtain the normal equations for x:

$$(A^TGA)x = A^TG(b_0 - a_0)$$

This is so, because the orthogonality of $V_A$ and $V_B$ implies

$$A^TGB = 0$$

Similarly we obtain the normals for y

$$(B^TGB)y = B^TG(a_0 - b_0)$$

11.3 Adjustment by minimizing the norm of the residuals.

Let L be the vector space of observations $\underline{l}$. Let $L_A$, $L_B$ be the orthocomplementary subspaces denoted $V_A$, $V_B$ earlier. Let $M_A$ be the manifold

$$M_A = \{\underline{l} \in L \mid \underline{l} = a_0 + a,\ a \in L_A\}$$

Here $a_0$ is a fixed vector in L. $M_A$ is called the manifold of adjusted observations. An observation vector in $M_A$ fulfills a number of geometrical or physical constraints, such as triangle closures or equations of motion.

The familiar problem of least squares adjustment searches for a vector $\tilde{\ell} \in M_A$ of adjusted observations. $\tilde{\ell}$ is chosen in a way that the (squared) norm of the residual vector

$$v = \tilde{\ell} - \ell$$

is minimized. Thus we have problem

(I) Given $\ell$, find $\tilde{\ell} \in M_A$ such that

$$d(\ell,\tilde{\ell})^2 = \text{Minimum}$$

In order to formulate the complementary problem, we need a manifold $M_B$ whose participating subspace is $L_B$, the orthocomplement of $L_A$. The vector $b_0$ must be chosen in a way that $\ell \in M_B$. The simplest choice is

$$b_0 = \ell$$

One could deviate from this simplest choice and reformulate problem (I) accordingly. However, we just take $b_0=\ell$. We now have problem

(II) Given $a_0$, find $\tilde{\ell} \in M_B$ such that

$$d(a_0,\tilde{\ell})^2 = \text{Minimum}$$

As we know, both problems have identical solutions. They are obtained as

$$\tilde{\ell} = a_0 + \Pi_A(\ell - a_0)$$
$$\tilde{\ell} = \ell + \Pi_B(a_0 - \ell)$$

If A, B are the matrices whose columns space $L_A$ and $L_B$, and if the inner product in L is represented by the weight matrix P, then we may proceed as follows

$$\tilde{\ell} = a_0 + Ax$$
$$\tilde{\ell} = \ell + Bk$$

Normal equations

$$(A^T P A)x = A^T P(\ell - a_0)$$
$$(B^T P B)k = B^T P(a_0 - \ell)$$

The optima fulfill:

$$d(\tilde{\ell}, a_0)^2 + d(\tilde{\ell}, \ell)^2 = d(a_0, \ell)^2$$

Lower and upper bounds for the optimum of (I) follow from

$$d(a_0, \ell)^2 - d(a_0, \ell'')^2 \leq d(\ell, \tilde{\ell})^2 \leq d(\ell, \ell')^2$$

provided that

$$\ell' \in M_A,\ \ell'' \in M_B$$

It remains to characterize the linear manifold $M_B$. It consists of all observation vectors $\ell''$ which result in the same adjusted values $\tilde{\ell}$ as $\ell$ does. It is even more instructive to characterize $L_B$. It consists of all observation vectors $b \in L$ whose adjustment results in the zero vector. (The vector of residuals is then the negative of the observation vector.) Hence problem (II) is formulated in words as follows:

(II): Find an observation vector of minimal norm which results in the same adjusted vector $\tilde{\ell}$ as the original vector $\ell$ does.

11.4 Adjustment by minimizing variances.

We take some time and space to review (in different notation) concepts introduced earlier (confer chapter B.4). The observations $\ell$ are now viewed as random variables. L is the space of their realizations, also called the sample space. The expectation $E(\ell) = \lambda$ is restricted to the submanifold $M_A$.

$$E(\ell) = \lambda \in M_A$$

The covariance matrix of the observations $\ell$ is

$$\Sigma(\ell) = Q\sigma^2$$

The positive definite matrix Q is known, the factor $\sigma^2$, called mean square unit weight error, is either known or unknown.

The inner product of L is represented by the weight matrix P. As pointed out in section 8.4.1, there is an isometry between L and its dual L'. The inner product in L' is represented by Q. Q is also called the reproducing kernel of L (not of L'). It holds that

$$Q = P^{-1}$$

The isometry is established by the representation of functionals as vectors. Confer sections 4.8 and 4.9! Let f be a functional $f \in L'$. If f is applied to a vector $\underline{l} \in L$ we write

$$f(\underline{l}) = f^T \underline{l} = f_1 \underline{l}_1 + \ldots + f_n \underline{l}_n$$

The representor r of f is a vector fulfilling

$$f(\underline{l}) = (r, \underline{l})$$

It follows that

$$f(\underline{l}) = f^T \underline{l} = r^T P \underline{l} = (r, \underline{l})$$

holds, if and only if

$$f = Pr$$

Equivalently

$$r = Qf$$

If f is represented by r and g by s, then

$$(f,g) = f^T Qg = (r,s) = r^T Ps$$

The isometry preserves the inner product.

Now let f be a functional out of L'. Writing

$$F(\ell) = f(\ell) + f_0 = f^T \ell + f_0 = \sum_{i=1}^{n} f_i \ell_i + f_0$$

this can also be viewed as a linear inhomogeneous function of the random variables $\ell$. As such $F(\ell)$ has the variance

$$\sigma^2(F(\ell)) = \sigma^2(f^T \ell) = f^T Qf \sigma^2$$

Up to the factor $\sigma^2$ this equals $\|f\|^2$, the squared norm of f.

Another linear inhomogeneous function $\hat{F}(\ell) = \hat{f}^T \ell + \hat{f}_0$ is called an unbiased estimator of f, provided that

$$E\{\hat{F}(\ell)\} = E\{F(\ell)\}$$

Whatever the value of $E(\ell) = \lambda \in M_A$ may be. We obtain

$$E\{\hat{f}^T \ell\} + \hat{f}_0 = E\{f^T \ell\} + f_0$$

$$\hat{f}^T E\{\ell\} + \hat{f}_0 = f^T E\{\ell\} + f_0$$

$$\hat{f}^T \lambda + \hat{f}_0 = f^T \lambda + f_0 \quad \text{for any } \lambda \in M_A$$

Replacing $\lambda$ by $\lambda = a_0 + a$, $a \in L_A$, it follows that

$$\hat{f}^T a = f^T a \quad \text{for any } a \in L_A, \quad \text{and} \quad \hat{f}^T a_0 + \hat{f}_0 = f^T a_0 + f_0$$

or

$$(f-\hat{f})^T a = 0 \quad \text{for any } a \in L_A, \quad \text{and} \quad \hat{f}_0 = (f-\hat{f})^T a_0 + f_0$$

Once, an unbiased functional $\hat{f}$ is established, the constant $f_0$ is easily determined from the last relation.

In practice $F(\ell)$ is related to a socalled "derived" quantity such as for example a distance between two remote points in a network. Replacing $F(\ell)$ by $\hat{F}(\ell)$ gives a random function $\hat{F}(\ell)$ having the same expectation as $F(\ell)$. One may exploit this fact trying to replace $F(\ell)$ by an $\tilde{F}(\ell)$ having a variance as small as possible. This optimal $\tilde{F}(\ell)$ will be called best unbiased estimator. Representing $\tilde{F}(\ell)$ as $\tilde{f}^T(\ell) + \tilde{f}_0$, the decisive problem is to find $\tilde{f}$. This problem is the following one:

Given $f \in L'$, find $\tilde{f} \in L'$ such that

$$(f-\tilde{f})a = 0 \quad \text{for any } a \in L_A$$

and

$$\|\tilde{f}\|^2 = \text{Minimum}$$

We further exploit the isometry between L and L'. The orthocomplementary subspaces $L_A$ and $L_B$ have their orthocomplementary counter-parts in L' as $L'_A$, $L'_B$. If $L_A$ and $L_B$ are spanned by the columns of A and B, then $L'_A$, $L'_B$ are spanned by the columns of A' and B'. Thereby

$$A' = PA \qquad A = QA'$$

$$B' = PB \qquad B = QB'$$

Functionals g fulfilling

$$(f-g)a = 0 \quad \text{for } a \in L_A$$

are recognized to be precisely functionals representable as

$$g = f + h \qquad h \in B'$$

The proof is obtained by using representing vectors:

$$h(a) = (b,a)$$

with b representing h. The inner product (b,a) is zero if and only if $b \in L_B$, i.e. $h \in L'_B$. We call the set of all functionals g fulfilling the above relation $M'_B$:

$$M_B' = \{g \in L' \mid g = f + h,\ h \in L_B'\}$$

$M_B'$ is the set of functionals $\hat{f}$ leading to unbiased estimators $\hat{F} = \hat{f}^T \ell + \hat{f}_0$ for $F(\ell)$ after choosing a suitable constant $\hat{f}_0$. Thus we arrive at the following problem, which we call

(II') Find $\tilde{f} \in M_B'$ having minimal norm.

It is seen that f plays the role of the vector $b_0$ in section 11.2, and that the zero vector now plays the role of the earlier vector $a_0$. Thus the complementary problem is immediately obtained by taking $M_A' = L_A'$.

(I') Given f find $\tilde{f} \in L_A'$ such that

$$\|f-\tilde{f}\|^2 = \text{Minimum}$$

We see that

$$\tilde{f} = \Pi_A' f$$
$$\tilde{f} = f + \Pi_B'(-f) = (I-\Pi_B')f$$

The projectors $\Pi_A'$, $\Pi_B'$ are represented by the matrices

$$\Pi_A' = A'(A'^T Q A)^{-1} A'^T Q$$
$$\Pi_B' = B'(B'^T Q B)^{-1} B'^T Q$$

Recalling A' = PA, B' = PB, one verifies

$$P'_A = P_A^T \ldots P_A = A(A^TPA)^{-1}A^TP$$
$$P'_B = P_B^T \ldots P_B = B(B^TPB)^{-1}B^TP$$

This brings out the fact that $\Pi'_A$ is the adjoint operator of $\Pi_A$:

$$(\Pi'_A f)(\ell) = f(\Pi_A \ell)$$

Similarly $\Pi'_B$ is the adjoint of $P_B$. This closes the gap between the two ways to perform an adjustment. An adjusted functional (a best linear unbiased estimator i.e. a BLUE) applied to a vector $\ell$ is the same as the original functional applied to the adjusted vector of observations.

Remark: A useful application of the complementary problems I' and II' is the a-priori specification of upper and lower bounds on the variance of the BLUE $\tilde{f}^T\ell$ of a linear function $f^T\ell$ of the observations. Any unbiased estimator $\hat{f}^T\ell$ will give an upper bound by II':

$$\sigma^2(\tilde{f}^T\ell) \leq \sigma^2(\hat{f}^T\ell)$$

On the other hand, by I', any functional $\overset{v}{f} \in L_A$, will give a lower bound

$$\sigma^2(\tilde{f}^T\ell) \geq \sigma^2(f^T\ell) - \sigma^2(\overset{v}{f}^T\ell)$$

If $L_A$ is spanned by the columns of A, then any functional in $L_{A'}$ is a linear combination of the columns of PA.

Thus, one can frequently specify useful upper and lower bounds on $\sigma^2(\tilde{f}^T\underline{\ell})$ before an adjustment is actually carried out.

## 12. Generalized inverses.

### 12.1. Range space and null space of a linear operator.

Let $V_m$ and $V_n$ be vector spaces of dimensions m and n respectively. We do not impose any restrictions such as $m \leq n$ onto the dimensions. Let $\Lambda$ be a linear operator from $V_m$ into $V_n$. After a choice of bases in $V_m$ and $V_n$, the operator is represented by an nxm matrix A.

The set of vectors $x \epsilon V_m$ which is mapped onto the zero vector forms a vector subspace $N(\Lambda)$ or briefly N of $V_m$. N is called null space of $\Lambda$.

$$N = N(\Lambda) = \{x \epsilon V_m \mid \Lambda(x) = 0\}$$

The set of vectors $y \epsilon V_n$, such that y is the image of some vector $x \epsilon V_m$, is called the range space $R(\Lambda)$ or briefly R. R is a vector subspace of $V_n$:

$$R = R(\Lambda) = \{y \epsilon V_n \mid y = \Lambda(x) \text{ for some } x \epsilon V_m\}$$

A basis of N is obtained by identifying a maximal linearly independent set of solutions x to the homogeneous system

$$Ax = 0$$

This can be accomplished by the Gauss-Jordan procedure (cf. section A.1.5).

A basis of R is given by a maximal number of linearly independent columns of A.

We know from section A.2.5 that an inverse operator $\Lambda^{-1}$ exists if and only if

$$m = n, \quad N = 0, \quad R = V_n$$

In this case the matrix A is nxn, its rank is n. The matrix is regular and possesses an inverse $A^{-1}$, the matrix representation of $\Lambda^{-1}$.

The theory of generalized inverses attempts to extend the notion of an inverse operator and an inverse matrix to situations where $\Lambda^{-1}$ and $A^{-1}$ no longer exist. Of course, some requirements of an inverse operator (inverse matrix) have to be relaxed.

12.2. The g-inverse.

A linear operator $\Lambda^g$ is called a g-inverse (generalized inverse), if it maps any $y \in R$ back onto a pre-image x of y:

$$\text{if } y \in R \text{ and } x = \Lambda^g(y) \text{ then } \Lambda(x) = y$$

This is equivalent to

$$\Lambda \circ \Lambda^g(y) = y \quad \text{for} \quad y \in R$$

Since any $y \in R$ may be represented as

$$y = \Lambda(x) \quad \text{for some } x \epsilon V_n$$

We also have

$$\Lambda \circ \Lambda^g \circ \Lambda(x) = \Lambda(x) \quad \text{for any } x \epsilon V_n$$

It follows that

$$\Lambda \circ \Lambda^g \circ \Lambda = \Lambda$$

is the necessary and sufficient requirement for $\Lambda^g$ to be a generalized inverse of $\Lambda$.

Let $A^g$ be the matrix representation of $\Lambda^g$. Note that $\Lambda^g$ maps $V_n$ into $V_m$. Hence $A^g$ is an mxn matrix, while A was nxm. We call $A^g$ a generalized inverse matrix of A. It is characterized by

$$A\, A^g\, A = A$$

The matrix $A^g$ is also characterized by the following property: If the system

$$Ax = y$$

is consistent (i.e. if y is in the span of the columns of A), then

$$x = A^g y$$

yields a solution of this system. There may be other solutions for the same right hand side y.

The operator $\Lambda^g$ and its matrix $A^g$ are generally not unique. This is plausible because the solution to a consistent system $Ax = y$ is generally not unique. Let $\theta_1$ be an operator from $V_n$ into $V_m$ whose range space $R(\theta_1)$ lies in $N = N(\Lambda)$. Let $\theta_2$ be an operator from $V_n$ into $V_m$ whose null space contains $R = R(\Lambda)$. If $\Lambda^g$ is any particular generalized inverse, then

$$\Lambda^g + \theta_1 + \theta_2$$

is also a generalized inverse. One readily verifies

$$\Lambda\circ(\Lambda^g + \theta_1 + \theta_2)\circ\Lambda = \Lambda$$

because $\Lambda\circ\theta_1 = 0$ and $\theta_2\circ\Lambda = 0$

<u>A general theorem on projectors.</u> Let $\Pi$ be an operator from a vector space V into itself. Necessary and sufficient for $\Pi$ to be a projector is the relation

$$\Pi\circ\Pi = \Pi$$

<u>Remark.</u> Let P be the matrix representation of $\Pi$. Then the above relation

presents itself as $PP = P$.

Definition. An operator (matrix) fulfilling $\Pi\circ\Pi = \Pi$ ($PP = P$) is called idempotent.

Proof of the theorem on projectors. Recall that a projector $\Pi$ induces a decomposition of $V$ into a direct sum of subspaces, the range space $R(\Pi)$ and the null space $N(\Pi)$. Vectors in the range space are reproduced: $\Pi(x) = x$ if $x = \Pi(y)$ for any $y$. Thus $\Pi(\Pi(y)) = \Pi(y)$ for any $y$. This proves $\Pi\circ\Pi = \Pi$, i.e. necessity. To prove sufficiency, assume that $\Pi\circ\Pi = \Pi$ holds. Consider $R(\Pi)$ and $N(\Pi)$. If $x\in R(\Pi)$, then $x = \Pi(y)$ for some $y$. From $\Pi\circ\Pi = \Pi$ we infer that $\Pi(x) = \Pi\circ\Pi(y) = \Pi(y) = x$. Thus vectors in $R(\Pi)$ are reproduced. Next we show that any vector $x$ can be represented as $x = x_1 + x_2$ with $x_1\in R(\Pi)$, $x_2\in N(\Pi)$. Just put $x_1 = \Pi(x)$ and $x_2 = x - \Pi(x)$. Then obviously $x_1\in R(\Pi)$ and $x_2\in N(\Pi)$. It remains to prove that $R(\Pi)$ and $N(\Pi)$ have only the zero vector in common. If $x\in R(\Pi)$ then $x = \Pi(x)$. If at the same time $x\in N(\Pi)$, then $\Pi(x) = 0$, i.e. $x = 0$. This was to be shown.

Remark. We are not talking about orthogonal projectors. An inner product may not even be defined in $V$.

Theorem. The operators $\Lambda\circ\Lambda^g$, $\Lambda^g\circ\Lambda$, represented by $AA^g$, $A^gA$, are projectors in $V_m$, $V_n$ respectively. It holds that

$$R(\Lambda\circ\Lambda^g) = R(\Lambda), \quad N(\Lambda^g\circ\Lambda) = N(\Lambda)$$

Proof. Both operators are verified to be idempotent. Hence they are projectors. Any vector y in $R(\Lambda)$ is represented as $\Lambda(x)$ for some x. From $\Lambda\circ\Lambda^g\circ\Lambda = \Lambda$, i.e. $\Lambda\circ\Lambda^g\circ\Lambda(x) = \Lambda(x)$ for any x, it follows that $\Lambda\circ\Lambda^g(y) = y$. Hence $R(\Lambda)$ includes $R(\Lambda\circ\Lambda^g)$. The reverse inclusion is trivial. Thus $R(\Lambda\circ\Lambda^g) = R(\Lambda)$. We turn to $\Lambda^g\circ\Lambda$. Obviously $N(\Lambda^g\circ\Lambda)$ includes $N(\Lambda)$. Suppose that there is a vector z in $N(\Lambda^g\circ\Lambda)$ which is not in $N(\Lambda)$. Then $(\Lambda^g\circ\Lambda)(z) = 0$, but $\Lambda(z) \neq 0$. Put these equations into matrix form

$$A^g(Az) = 0, \quad Az \neq 0$$

This tells us that x=0 is a solution to the consistent system

$$Ax = Az, \quad Az \in R(\Lambda), \quad Az \neq 0$$

This is impossible. Hence $N(\Lambda^g\circ\Lambda) = N(\Lambda)$, as was to be shown.

Remark. If $\Lambda^{-1}$ is the ordinary inverse of $\Lambda$, then $\Lambda$ is the ordinary inverse of $\Lambda^{-1}$. This is generally not true in the case of $\Lambda$ and $\Lambda^g$: $\Lambda$ may not be a $(\Lambda^g)^g$. This failure will be repaired in the next subsection by imposing further restrictions on $\Lambda^g$.

12.3. Reflexive generalized inverse. In addition to

$$\Lambda\circ\Lambda^g\circ\Lambda = \Lambda, \quad \text{i.e.} \quad AA^gA = A$$

we also require

$$\Lambda^g \circ \Lambda \circ \Lambda^g = \Lambda^g, \text{ i.e. } A^g A A^g = A^g$$

Then $\Lambda$ is also a generalized inverse of $\Lambda^g$. The roles of $\Lambda$ and $\Lambda^g$ can be interchanged. We call such a generalized inverse "reflexive". It will be denoted by $\Lambda^r$. Its matrix representation is $A^r$. We repeat the above equations in the new notation:

$$\Lambda \circ \Lambda^r \circ \Lambda = \Lambda, \text{ i.e. } AA^rA = A$$
$$\Lambda^r \circ \Lambda \circ \Lambda^r = \Lambda^r, \text{ i.e. } A^rAA^r = A$$

It follows that the projectors $\Lambda \circ \Lambda^r$ and $\Lambda^r \circ \Lambda$ fulfill:

$$R(\Lambda \circ \Lambda^r) = R(\Lambda), \quad N(\Lambda \circ \Lambda^r) = N(\Lambda^r)$$
$$R(\Lambda^r \circ \Lambda) = R(\Lambda^r), \quad N(\Lambda^r \circ \Lambda) = N(\Lambda)$$

From these equations and a dimension argument one can infer that A and $A^g$ have equal rank. One can also show that this requirement is sufficient for $A^r$ to be a reflexive inverse.

12.4. Generalized inverse with least squares property.

Assume an inner product in $V_n$. Let it be represented by the matrix $G_n$. We may form the orthocomplement $R^\perp$ of R. We consider a generalized inverse $\Lambda^g$, i.e. we require

$$\Lambda \circ \Lambda^g \circ \Lambda = \Lambda, \quad \text{i.e.} \quad AA^gA = A$$

In addition we postulate

$$N(\Lambda \circ \Lambda^g) = R^\perp, \quad \text{i.e.} \quad AA^gy = 0 \text{ is equivalent to } y \in R^\perp.$$

The projector $\Lambda \circ \Lambda^g$ is thus required to be an orthogonal projector. We denote such a generalized inverse by $\Lambda^l$, and its matrix by $A^l$. The importance of $A^l$ is stressed in the following

Theorem. Consider the (generally) inconsistent system

$$Ax = y$$

Let y be arbitrary. Then

$$x = A^gy$$

fulfills the least squares reqirement

$$\underset{z \in V_m}{\text{Min}} \|y - Az\| = \|y - Ax\|$$

if and only if $A^g = A^l$.

Proof. Given $y \epsilon V_n$, decompose it as

$$y = y_1 + y_2, \quad y_1 \epsilon R, \quad y_2 \epsilon R^{\perp}$$

Suppose that $N(\Lambda \circ \Lambda^g) = R^{\perp}$, then the projector $\Pi = \Lambda \circ \Lambda^g$ onto R is an orthogonal projector. Calculate

$$x = \Lambda^g(y)$$

then

$$z = \Lambda(x) = \Lambda \circ \Lambda^g(y) = \Pi(y)$$

is the orthogonal projection of y onto R. Thus z is the solution of the stated extremum problem.

Suppose now that

$$x = \Lambda^g(y)$$

gives the least squares solution for any y. Then

$$\Lambda(x) = \Lambda \circ \Lambda^g(y)$$

must be the orthogonal projection of y onto R. Thus $\Lambda \circ \Lambda^g$ is the orthogonal

projector onto R.

A general theorem on projectors.

Let V be an inner product space. Necessary and sufficient for an operator $\Pi$ to be an orthogonal projector is

$$\Pi \circ \Pi = \Pi \quad \text{and} \quad \Pi^* = \Pi$$

Remark. $\Pi^*$ is the adjoint operator of $\Pi$ in the sense of section A.4.10, i.e. it holds that $(\Pi(x),y) = (x,\Pi^*(y))$. If the inner product in V is represented by the matrix G, and if $\Pi$ is represented by P, then the above conditions are restated as

$$PP = P$$
$$P^* = P, \quad P^* = G^{-1}P^TG$$

Proof. It was shown above that $\Pi \circ \Pi = \Pi$ is necessary and sufficient for $\Pi$ to be a projector onto $R(\Pi)$, and that V is spanned by $R(\Pi)$ and $N(\Pi)$. It suffices to show that $\Pi = \Pi^*$ is equivalent to $R(\Pi)^\perp = N(\Pi)$. Assume that $\Pi = \Pi^*$. From the defining equation for the adjoint operator [which is $(\Pi(x),y) = (x,\Pi^*(y))$], we get

$$(\Pi(x),y) = (x,\Pi(y))$$

This shows: If $y \in N(\Pi)$ and $x \in R(\Pi)$ then $0 = (x,0) = (x,\Pi(y)) = (\Pi(x),y) = (x,y) = 0$. Thus $N(\Pi)$ is included in $R(\Pi)^\perp$.

If x is arbitrary and $y \in R(\Pi)$, then $(x,\Pi(y)) = (\Pi(x),y) = 0$. This shows that $\Pi(y) = 0$. Hence $R(\Pi)$ is included in $N(\Pi)$. Thus $\Pi = \Pi^*$ indeed implies $R(\Pi) = N(\Pi)$.

Now assume $R(\Pi) = N(\Pi)$, i.e. assume that $\Pi$ is an orthogonal projector. It was shown in section A.5.6 that $\Pi = \Pi^*$. We give another proof as follows. From

$$(\Pi^*(x),y) = (x,\Pi(y))$$

we deduce for <u>any</u> x:

(1) if $y \in R(\Pi)$: $(\Pi^*(x),y) = (x,\Pi(y)) = (x,y) = (\Pi(x) + (I-\Pi)(x),y) = (\Pi(x),y)$

(2) if $y \in R(\Pi)^\perp$: $(\Pi^*(x),y) = (x,\Pi(y)) = (x,0) = 0 = (\Pi(x),y)$

Thus $(\Pi^*(x),y) = (\Pi(x),y)$ holds for all x and all y. Hence $\Pi^* = \Pi$.

Using the theorem we arrive at the following characterization of a least squares inverse $\Lambda^!$ represented by $A^!$:

$$\Lambda \circ \Lambda^! \circ \Lambda = \Lambda \qquad \text{or } AA^!A = A$$
$$(\Lambda \circ \Lambda^!)^* = \Lambda \circ \Lambda^! \quad \text{or } G_n^{-1}(AA^!)^T G_n = AA^!$$

<u>12.5. Generalized inverse with minimum norm property.</u>

Suppose that $V_m$ is equipped with an inner product. $V_n$ does not necessarily have an inner product. In addition to

$$\Lambda \circ \Lambda^g \circ \Lambda = \Lambda$$

we require

$$R(\Lambda^g \circ \Lambda) = N$$

Because $N(\Lambda^g \circ \Lambda) = N$, we thus require that $\Lambda^g \circ \Lambda$ is an orthogonal projector. We call a g-inverse fulfilling these conditions a minimum norm inverse $\Lambda^m$. Its matrix is denoted by $A^m$.

Theorem. Suppose that the system

$$Ax = y$$

is consistent. Otherwise let y be arbitrary. Then

$$x = A^g y$$

fulfills the minimum norm requirement

$$\min_{\substack{z \in V_m \\ Az = y}} \|z\| = \|x\|$$

if and only if $A^g = A^m$.

Proof. Consider the consistent system

$$Az = y$$

Represent z as

$$z = z_1 + z_2, \quad z_1 \epsilon N^{\perp}, \; z_2 \epsilon N$$

All solutions to the system Az = y are obtained by keeping $z_1$ fixed (as a particular solution to the homogeneous system) and by letting $z_2$ vary over N (as the general solution to the homogeneous system Az = 0). Because $\|z\|^2 = \|z_1\|^2 + \|z_2\|^2$, the minimal solution is obviously $z = z_1$. It is obtained as $z_1 = A^g y$ if and only if the image of $A^g y$ is in $N^{\perp}$ for any $y \epsilon R$. Such y are represented as y = Au. Thus $A^g A$ u must be in $N^{\perp}$ for any u. Because $A^g A$ is a projector whose null space is N, $A^g A$ must be the projector onto $N^{\perp}$, i.e. it must be an orthogonal projector.

The characterization of a minimum norm inverse is

$$\Lambda \circ \Lambda^m \circ \Lambda = \Lambda \qquad \text{or } AA^mA = A$$
$$(\Lambda^m \circ \Lambda)^* = \Lambda^m \circ \Lambda \text{ or } (A^mA)^* = A^mA \text{ with } (A^mA)^* = G_m^{-1}(A^mA)^T G_m$$

12.6. The minimum norm least squares inverse.

Assume that an inner product is specified in $V_m$ as well as in $V_n$. Given an

operator $\Lambda$ from $V_m$ into $V_n$, we are searching for a $\Lambda^g$ giving a least squares solution of minimal norm. We shall call such an inverse $\Lambda^{lm}$. Translated into the language of matrices, we start from a (generally) inconsistent system

$$Ax = y$$

and we search for x fulfilling

$$\|x\| = \min_{z \in Z} \|z\|$$

where Z is the set of least squares solutions defined by

$$Z = \{z \in V_m \mid \|y - Az\| \leq \|y - Au\| \text{ for any } u \in V_m\}$$

We shall show that the solution to this problem is unique. Thus $\Lambda^{lm}$ and $A^{lm}$ will be unique. We decompose $V_m$ and $V_n$ into direct sums:

$$V_m = N^{\perp} + N$$
$$V_n = R + R^{\perp}$$

Decomposing y as

$$y = y_1 + y_2, \quad y_1 \in R, \quad y_2 \in R^{\perp}$$

the set Z of least squares solutions z is given by

$$Az = y_1$$

[If $Az = y_1 + \eta_1$ with $\eta_1 \epsilon R$, then we would have $\|y - Az\|^2 = \|\eta_1\|^2 + \|y_2\|^2$. Obviously this is minimal for $\eta_1 = 0$.] We decompose

$$z = z_1 + z_2, \quad z_1 \epsilon N^{\perp}, \quad z_2 \epsilon N$$

Because

$$\|z\|^2 = \|z_1\|^2 + \|z_2\|^2$$

the minimal z is obviously $z = z_1$. This concludes the proof. From the proof it is clear that

(1) The operator $\Lambda$ maps $N^{\perp}$ onto R and N onto 0

(2) The operator $\Lambda^{lm}$ maps R back onto $N^{\perp}$ and $R^{\perp}$ onto 0.

If the operators $\Lambda$, $\Lambda^{lm}$ are restricted to the subspaces $N^{\perp}$ and R, then $\Lambda^{lm}$ is the conventional inverse of $\Lambda$.

The operator $\Lambda^{ln}$ and its matrix $A^{lm}$ may be uniquely characterized by the following equations:

(1) $A \circ A^{lm} \circ A = A$ or $AA^{lm}A = A$

(2) $A^{lm} \circ A \circ A^{lm} = A^{lm}$ or $A^{lm}AA^{lm} = A^{lm}$

(3) $(A \circ A^{lm})^* = A \circ A^{lm}$ or $(AA^{lm})^* = AA^{lm}$

(4) $(A^{lm} \circ A)^* = A^{lm} \circ A$ or $(A^{lm}A)^* = A^{lm}A$

The necessity of these relations is easily proved: (1) holds for any $A^g$, (3) holds for a least squares inverse, (4) holds for a minimum norm inverse. (2) is proved directly from the above geometric characterization of $A$ and $A^{lm}$. (Just verify what $A^{lm}$ does to vectors in $R$ and $R^\perp$).

Sufficiency of (1), (3), (4) is also clear from earlier sections. The sufficiency of (2) is an interesting question. Note that a minimum norm inverse, as considered in section 12.4, guarantees a minimum norm solution only for a consistent system. However, in this section we want minimum norm solutions also for inconsistent systems. Here condition (2) steps in, excluding inverses which would not give minimum solutions to inconsistent systems. We do not further elaborate but leave it with the hint that one must be concerned with the images of vectors in $R^\perp$ under $A^g$.

## 12.7. The pseudo inverse.

Assume that the inner products in $V_m$, $V_n$ are represented by the identity matrix (of appropriate dimensions m and n). The matrix representing the minimum norm least squares inverse is then denoted by $A^+$. It is called the pseudo inverse of A. Sometimes also the name "Moore-Penrose inverse" is used. $A^+$ is unique and uniquely characterized by the following relations

(1) $AA^+A = A$

(2) $A^+AA^+ = A^+$

(3) $(AA^+)^T = AA^+$

(4) $(A^+A)^T = A^+A$

Thus the pseudo inverse represents the minimum norm least squares inverse in case of orthonormal bases in $V_m$ and $V_n$.

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## 13. Adjustment of rank deficient systems.

### 13.1. Formulation of the problem.

We introduce the following spaces

L ... n-dimensional space of observations (sample space)

$L_A$ ... r-dimensional subspace of adjusted observations

X ... m-dimensional parameter space, $m \geq r$

There is a mapping $\Lambda$ from X onto $L_A$. It is represented by the nxm matrix A. The mapping is not unique if $m > r$. The rank of A is r.

The inner product in L is represented by the weight matrix P. The inner product in X shall be represented by G.

The adjustment problem is formulated as follows. Given a vector $\ell \in L$ of observations, find corrections $v \in L$ and parameters $x \in X$ such that

$$\ell + v = A\,x$$

$$v^T P v = \text{minimum}$$

The solution $\tilde{x}$ for x is generally not unique. However, the corrections are unique.

### 13.2. Solution via generalized inverses of A.

Let $A^!$ be a least squares inverse of A. Then

$$\tilde{x} = A^{\mathbf{l}}\mathbf{l}$$

is a solution of the adjustment problem. This is the statement of the theorem in section 12.4.

Let $A^{\mathbf{lm}}$ be the (unique) minimum norm least squares inverse of A, then

$$\tilde{x} = A^{\mathbf{lm}}\mathbf{l}$$

is the solution of the least squares problem having minimal norm. This is the result of section 12.6.

13.3. A minimum property of the covariance of the adjusted parameters $\tilde{x} = A^{\mathbf{ln}}\mathbf{l}$.

Using any least squares inverse $A^{\mathbf{l}}$, the covariance of the adjusted parameters $\tilde{x}$ is

$$\Sigma(\tilde{x}) = A^{\mathbf{l}}P^{-1}(A^{\mathbf{l}})^{T}\sigma^2$$

Let $M = (m_{ij})$ be an nxn matrix. The trace of M is defined as

$$tr(M) = \sum_{i=1}^{n} m_{ii}$$

The following facts about the trace of a matrix are needed in the subsequent theorem:

(*) if M is positive semidefinite, then $\mathrm{tr}(M) \geq 0$. (This is clear, because a positive semidefinite matrix has nonnegative diagonal elements. See section A.4.5 for the definition of positive definite matrices.)

(*) $\mathrm{tr}(MN) = \mathrm{tr}(NM)$. The proof is an easy exercise. Note that M and N need not be square matrices, only MN must be square.

Theorem. It holds that

$$\mathrm{tr}\{GA^{lm} P^{-1} (A^{lm})^T\} \leq \{GA^{l} P^{-1} (A^{l})^T\}$$

Thus the trace of $G\,\Sigma(\tilde{x})$ is minimal if $A^l$ is chosen as $A^{lm}$.

Proof. Let $x = A^l y$. Decompose $x = x_1 + x_2$ with $x_1 \in N^\perp$ and $x_2 \in N$ ($N = N(A)$ being the null space of A). Then $x_1 = A^{lm} y$. We call $B = A^l - A^{lm}$. The range $R(B)$ is in N. Thus $B^T G A^{lm} = 0$ holds (recall that G represents the inner product in X). Next observe that

$$\begin{aligned}
&\mathrm{tr}\{G A^l P^{-1} (A^l)^T\} = \\
&= \mathrm{tr}\{G [A^{lm} + B] P^{-1} [A^{lm} + B]^T\} = \\
&= \mathrm{tr}\{G A^{lm} P^{-1} (A^{lm})^T\} + \mathrm{tr}\{G A^{lm} P^{-1} B^T\} + \\
&\quad \mathrm{tr}\{G B P^{-1} (A^{lm})^T\} + \mathrm{tr}\{G B P^{-1} B^T\} = \\
&= \mathrm{tr}\{G A^{lm} P^{-1} (A^{lm})^T\} + \mathrm{tr}\{G B P^{-1} B^T\}
\end{aligned}$$

because $tr\{G\, A^{lm}\, P^{-1}\, B^T\} = tr\{B^T\, G\, A^{lm}\, P^{-1}\} = 0$, since $B^T G\, A^{lm} = 0$. Similarly $tr\{G\, B\, P^{-1}\, (A^{lm})^T\} = 0$.

Now we focus attention on the trace

$$tr\{G\, B\, P^{-1}\, B^T\}$$

Decomposing $G = RR^T$ into Cholesky factors, we have

$$tr\{G\, B\, P^{-1}\, B^T\} = tr\{R\, B\, P^{-1}\, B^T R^T\}$$

Now $R\, B\, P^{-1}\, B^T R^T = (RB)\, P^{-1}\, (RB)^T$ is positive semidefinite. (If M is positive semidefinite, so is $WMW^T$ for any W. The proof of this just uses the definition of positive semidefiniteness: For arbitrary x we have $x^T\, (WMW^T)\, x = (W^T x)^T\, M \cdot (W^T x) = y^T My$, with $y = W^T x$. However, $y^T My > 0$, because M is positive semidefinite.) Thus $tr\{R\, B\, P^{-1}\, B^T R^T\} = tr\{G\, B\, P^{-1}\, B^T\} > 0$ and the theorem is proved.

13.4. Solution via singular normal equations.

We form the normal equations

$$(A^T P\, A)\, \tilde{x} = A^T P l$$

They are singular if m>r. Any solution to these normals is a least squares solution. This follows from the fact that the normals require nothing else but

the orthogonality of $v = A\tilde{x} - \ell$ and the columns of A. The normal equations are always consistent, because the projection of $\ell$ onto the space spanned by the columns of A must exist. (An explicit consistency proof is given in section 13.6.) Thus, if $(A^TPA)^g$ is any generalized inverse of $A^TPA$, then

$$\tilde{x} = (A^TPA)^g A^TP\ell = A^{l}\ell$$

It follows that

$$A^{l} = (A^TPA)^g A^TP$$

This equation means that if $(A^TPA)^g$ runs through all generalized inverses of $A^TPA$ then, in any case, a least squares inverse $A^{l}$ is obtained. If $(A^TPA)^m$ is any minimum norm inverse, then

$$\tilde{x} = (A^TPA)^m A^TP\ell$$

must be the least squares solution having minimal norm. Thus

$$A^{lm} = (A^TPA)^m A^TP$$

We find for $\tilde{x} = A^{lm}\ell$:

$$\Sigma(\tilde{x}) = (A^TPA)^m A^TPP^{-1}PA (A^TPA)^m \sigma^2 =$$
$$= (A^TPA)^m (A^TPA) (A^TPA)^m \sigma^2$$

If $(A^TPA)^m$ is chosen as $(A^TPA)^{\ell m}$, then, due to the reflexivity of $(A^TPA)^{\ell m}$, we have

$$\Sigma(\tilde{x}) = (A^TPA)^{\ell m}\ \sigma^2$$

This, by the way, shows

$$(A^TPA)^{\ell m} = (A^TPA)^m\ (A^TPA)\ (A^TPA)^m$$

More important is the minimum trace property derived in section 13.3:

$\Sigma(\tilde{x})$ is minimal if $(A^TPA)^m$ is chosen as $(A^TPA)^{\ell m}$

## 13.5. Calculation of the $\ell m$-inverse $A^{\ell m}$.

There are many ways to calculate the unique inverse $A^{\ell m}$. We consider two procedures. The first one is recommended if rank(A) is small compared to the size of the nxm matrix A, more precisely if

$$\text{rank}(A) \ll \min(n,m)$$

The second procedure works well for matrices which are nearly square and whose rank is nearly equal to the size, more precisely

$$\text{rank}(A) \doteq m \doteq n$$

Method (1). Let r be the rank of A. Consider a rank factorization of A, i.e. a decomposition

$$A = BC^T$$

where B ... nxr and $C^T$...rxm are both of rank m. Then

$$A^{\dagger m} = GC\,(C^TGC)^{-1}\,(B^TPB)^{-1}\,B^TP$$

The proof follows by verifying the four conditions for $A^{\dagger m}$ given in section 12.6. (Recall that G represents the inner product in the domain space of A.)

Remark. A rank factorization for A may be deduced from the last stage of the Gauss-Jordan procedure (exercise).

Method (2). Consider matrices S, N of size n x (n-r) and m x (m-r), such that S spans R(A) and N spans N(A):

$$A^TPS = 0$$
$$AN = 0$$

Both S and N have the maximal number of linearly independent columns fulfilling the two homogeneous systems above.

Consider the square matrix of size (n+m-r) x (n+m-r)

$$H = \begin{bmatrix} A & S \\ N^T G & 0 \end{bmatrix}$$

We show that this matrix is regular by showing that Hz = 0 implies z = 0.

Decompose z as

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

Then Hz = 0 means

$$Ax + Sy = 0$$
$$N^T Gx = 0$$

The spaces spanned by the columns of A and S are orthocomplementary. Hence Ax = 0 and Sy = 0 must hold. Since S has linearly independent columns, we infer y = 0. We are left with

$$Ax = 0$$
$$N^T Gx = 0$$

Such a vector x must be in N(A) and, at the same time, it must be orthogonal to N(A) which is spanned by the columns of N. Thus x = 0 and H is indeed regular.

We form the inverse of H, and we denote it as

$$H^{-1} = \begin{bmatrix} Q & K \\ L^T P & M \end{bmatrix}$$

Our aim is to show that $Q = A^{\#m}$. As a first step we show that $M = 0$. From $HH^{-1} = I$ we deduce

$$AK + SM = 0$$

In the regularity proof for H we have seen that these equations imply $M = 0$. Thus

$$H^{-1} = \begin{bmatrix} Q & K \\ L^T P & 0 \end{bmatrix}$$

We now write out the equations for $HH^{-1} = I$ in full:

$$AQ + SL^T P = I \quad (a)$$
$$AK = 0 \quad (b)$$
$$N^T GQ = 0 \quad (c)$$
$$N^T GK = I \quad (d)$$
$$QA + KN^T G = I \quad (a')$$
$$QS = 0 \quad (b')$$
$$L^T PA = 0 \quad (c')$$
$$L^T PS = I \quad (d')$$

Post-multiplying (a) by A and minding (c') we find the first condition of

section 12.6 for a minimum norm least square inverse Q

$$AQA = A \quad (1)$$

Post-multiplying (a') by Q and minding (c) we get

$$QAQ = Q \quad (2)$$

We see that Q is a reflexive inverse of A. Thus

$$\text{rank}(Q) = \text{rank}(A) = r$$

We are done if we can show that the projectors AQ and QA are orthogonal projectors. This is equivalent to conditions (3) and (4) for $A^{lm}$ as given in section 12.6:

$$(AQ)^* = AQ \quad (3)$$
$$(QA)^* = QA \quad (4)$$

What we need to show is that the null space of AQ is S and that the range space of QA is $N^\perp$. The first assertion may be deduced from (b'), the second one from (c). In both cases it is necessary to observe that the rank of AQ and QA is r. This implies that the null space of AQ cannot be larger than the space spanned by S, and that the range space of QA cannot be larger than the orthocomplement of N. Thus we have proved that the submatrix Q of $H^{-1}$ is indeed identical to $A^{lm}$.

Remark. From A being the $Q^{lm}$-inverse of Q, and from reasons of symmetry, we may deduce that the submatrices K and L of $H^{-1}$ span $R(Q)^{\perp}$ and N(Q). This also follows algebraically from $R(Q)^{\perp} = N(A)$ together with (b), and from $N(Q) = R(A)^{\perp}$ together with (c'). Thus $R(A)^{\perp} = N(Q)$ is spanned by S as well as by L, and $N(A) = R(Q)^{\perp}$ is spanned by N as well as by K. Confer the geometric characterization of $A^{lm}$ given in section 12.6.

13.6. Application to free network adjustment.

We illustrate the principle by assuming, as a special example, a network in the plane involving distance measurements. There may also be a number of measured angles or unoriented directions. However, no azimuth measurements shall be available. Also measurements of absolute positions shall be absent. The coordinates of the network points are denoted by $x_i$, i=1,...,n. As usual they are represented as

$$x_i = x_i^{(0)} + \Delta x_i, \quad y_i = y_i^{(0)} + \Delta y_i, \quad i=1,\ldots,n$$

whereby $x_i^{(0)}$, $y_i^{(0)}$ denote known approximate values and $\Delta x_i$, $\Delta y_i$ denote small unknown increments. We introduce the vector $\Delta z$ by $\Delta z^T = (\Delta x_1, \Delta y_1, \Delta x_2, \Delta y_2, \ldots .. \Delta x_n, \Delta y_n)^T$. We do not assign fixed coordinates to any of the network points, nor do we fix any of the azimuths within the network. As a result the observation equations

$$\Delta\ell + v = A\,\Delta z$$

are singular, i.e. the linear system

$$A\,\Delta z = 0$$

has nonzero solutions. If we form the normal equations

$$(A^T P A)\,\Delta z = A^T P\,\Delta\ell$$

then also these equations are singular. The system

$$(A^T P A)\,\Delta z = 0$$

has nonzero solutions. It is not too difficult to show that the solutions for $A\,\Delta z = 0$ and $(A^T P A)\,\Delta z = 0$ coincide, i.e. that the null space of A equals the null space of $A^T P A$. (If $A\,\Delta z = 0$ then trivially $A^T P A\,\Delta z = 0$. If $A\,\Delta z \neq 0$ then $(A\,\Delta z)^T P (A\,\Delta z) > 0$, since P is positive definite. It follows that $\Delta z^T (A^T P A)\,\Delta z > 0$, implying $(A^T P A)\,\Delta z \neq 0$. Q.e.d.)

On the other hand, the normal equations are consistent, i.e. a solution $\Delta z$ fulfilling $(A^T P A)\,\Delta z = A^T P \Delta\ell$ exists for any choice of $\Delta\ell$. For a proof start from $A\,\Delta z = \Delta\ell + v$, decompose $\Delta\ell$ into $\Delta\ell_1 + \Delta\ell_2$ with $\Delta\ell_1 \in R(A)$ and $\Delta\ell_2 \in R(A)^\perp$. Choose $v = -\Delta\ell_2$. Then $A\,\Delta z = \Delta\ell_1$ is consistent; so is $(A^T P A)\,\Delta z = (A^T P)\,\Delta\ell_1 = A^T P\,\Delta\ell$.

The proof just given demonstrates that any solution to the normal equations gives a least squares solution to the generally inconsistent observation equations

$$A\,\Delta z = \Delta \ell$$

However, the solution to the normal equations is not unique. All kinds of solutions are obtained if we write

$$\Delta z = (A^T P A)^g\, A^T P\, \Delta \ell$$

where $(A^T P A)^g$ is any generalized inverse of the normal equation matrix $A^T P A$.

The various solutions $\Delta z$ obtained in this way differ by solutions $\Delta z_n$ to the homogeneous system

$$(A^T P A)\, \Delta z_n = 0$$

or also

$$A\, \Delta z_n = 0$$

These solutions are easy to specify from geometrical considerations. Since

$$\Delta \ell = A \, \Delta z$$

are the changes of the observables $\ell$ if the coordinates are changed by $\Delta z$, we must look for such coordinate changes which leave the observables unchanged. Such coordinate changes are implied by a translation and a rotation of the whole set of points.

If the whole set of points is translated by $\Delta c_x$, $\Delta c_y$ and rotated by a small angle $\Delta\varphi$, then the coordinate changes are given by

$$\Delta z = \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \\ . & . & . \\ . & . & . \\ . & . & . \\ 1 & 0 & -y_n \\ 0 & 1 & x_n \end{bmatrix} \begin{bmatrix} \Delta c_x \\ \Delta c_y \\ \Delta\varphi \end{bmatrix}$$

$$= N \, \Delta t, \text{ say}$$

It follows that

$$\Delta \ell = A \, N \, \Delta t = 0, \quad \text{i.e.} \quad AN = 0$$

The three columns of N span the null space of the matrix A. As we know the null space of A coincides with that one of $A^TPA$.

We assume an inner product in the parameter space implied by the unit matrix: G = I. Then the ℓm-inverse of $A^TPA$ reduces to the pseudo inverse $(A^TPA)^+$. Applying the theorem of section 13.5, we get this pseudo inverse by taking the appropriate submatrix of

$$\begin{bmatrix} A^TPA & N \\ N^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} (A^TPA)^+ & K \\ K^T & 0 \end{bmatrix}$$

In this case

$$\Sigma(\tilde{x}) = (A^TPA)^+ \sigma^2$$

has minimal trace among all covariance matrices $\Sigma(\tilde{x})$ resulting from all choices of least squares solutions to the rank deficient network adjustment problem.

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