Dealing with stochastic properties of large data sets

Wolf-Dieter Schuh, Jan Martin Brockmann, Lukas Jendges, Johannes Korte, Till Schubert

Institute of Geodesy and Geoinformation
Theoretical Geodesy Group
University Bonn

COLLECTOR Workshop
Uncertainty quantification of point clouds

Fulda, December 4, 2023

Goal of this presentation

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

The goal of this lecture is to comment on the following points :

- Is it necessary to model covariances in large datasets at all?
- How to efficiently handle stochastic properties of large data sets? What methods are available?
- How and when to apply these methods?

This lecture can only be an introduction and appetizer and start for a discussion!

Goal of this presentation

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

The goal of this lecture is to comment on the following points :

- Is it necessary to model covariances in large datasets at all?
- How to efficiently handle stochastic properties of large data sets? What methods are available?
- How and when to apply these methods?

This lecture can only be an introduction and appetizer and start for a discussion!

Further Readings:

Cressie (1991),
Brockwell and Davis (2002),
Matérn (1960),
...

Isaaks and Shrivastava (1989),
Wackernagel (2003),
Schuh (2016),
...

Contents

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Motivation
- Least Squares Collocation
  - Filter Approach
  - Covariance Approach
- Further reflections — tailored systems
- Example: GOCE time series - prewhitening
- Example: DInSAR point cloud - signal prediction
- Summary and Discussion

Motivation
measurement model
parameterization
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Motivation

Measurement model

Motivation
measurement model
parameterization
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$f(x)$ ... deterministic model
(x ... parameter)

s ... additional signal

n ... white noise

measurements: ℓ ... single realization

$$\ell = f(x) + s + n$$

x, s, n ... single realizations

random variables: $\mathcal{L}$ ... stochastic process

$$\mathcal{L} = f(\xi) + \mathcal{S} + \mathcal{N}$$

$\mathcal{S}, \mathcal{N}$ ... stochastic processes
ξ ... true value

Measurement model

Motivation
measurement model
parameterization
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\mathcal{L} = f(\xi, x_{space}, x_{time}) + \mathcal{S}(x_{space}, x_{time}) + \mathcal{N}$$

Parameterization of the signal

Motivation
measurement model
parameterization
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\boxed{\mathcal{L} = f(x, x_s, x_t) + \mathcal{S}(x_s, x_t) + \mathcal{N}}$$

Parameterization of the signal

Motivation
measurement model
parameterization
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\mathcal{L} = f(x, x_s, x_t) + \mathcal{S}(x_s, x_t) + \mathcal{N}$$

- **external prior information**

  $\Delta\mathcal{L} = f(x, x_s, x_t) + \mathcal{N}$ $\qquad$ $\Delta\mathcal{L} = \mathcal{L} - s(x_s^{(p)}, x_t^{(p)})$

  - calibration
  - reduction strategies
  - de-aliasing

Parameterization of the signal

Motivation
measurement model
parameterization
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\mathcal{L} = f(x, x_s, x_t) + \mathcal{S}(x_s, x_t) + \mathcal{N}$$

- **external prior information**
  $\Delta\mathcal{L} = f(x, x_s, x_t) + \mathcal{N}$ $\qquad$ $\Delta\mathcal{L} = \mathcal{L} - s(x_s^{(p)}, x_t^{(p)})$
- **deterministic parameterization** - **non-parametric models**
  $\mathcal{L} = A_1 x_1 + A_2 x_2 + \mathcal{N}$
  - linear combination of base functions (polynoms, splines, finite elements)
  - co-estimation (Schur-form, projection method)
  - empirical parameters, stochastic impulses
  - nuisance variables, short arc approach

Parameterization of the signal

Motivation
measurement model
parameterization
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\mathcal{L} = f(x, x_s, x_t) + \mathcal{S}(x_s, x_t) + \mathcal{N}$$

- **external prior information**
  $\Delta\mathcal{L} = f(x, x_s, x_t) + \mathcal{N}$ $\qquad$ $\Delta\mathcal{L} = \mathcal{L} - s(x_s^{(p)}, x_t^{(p)})$
- **deterministic parameterization** - **non-parametric models**
  $\mathcal{L} = A_1 x_1 + A_2 x_2 + \mathcal{N}$
- **stochastic parameterization** - $\mathcal{S}(x_s, x_t)$ **...random process**
  $\mathcal{L} = f(x, x_s, x_t) + \mathcal{S}(x_s, x_t) + \mathcal{N}$ $\qquad$ $E\{\mathcal{S}\}, \Sigma\{\mathcal{S}\}$
  - Prediction, Collocation - Kernel functions
  - Wiener-Kolmogorov filtering - Covariance functions
  - Kriging - Variograms
  - Frequency selective metric - Linear discrete filters

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Least Squares Collocation (LSC)

Least Squares Collocation - BLUP

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\boxed{\begin{aligned} \boldsymbol{\mathcal{L}} = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}} \qquad & \boldsymbol{E}\{\boldsymbol{\mathcal{S}}\} \stackrel{!}{=} \mathbf{0}_n, \quad \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}\} \\ & \boldsymbol{E}\{\boldsymbol{\mathcal{N}}\} \stackrel{!}{=} \mathbf{0}_n, \quad \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{N}}\} \\ & \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{N}}\} \stackrel{!}{=} \mathbb{O}_n \end{aligned}}$$

$$\boldsymbol{\Sigma} := \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}}\}$$

Least Squares Collocation - BLUP

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\boxed{\begin{aligned} \boldsymbol{\mathcal{L}} = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}} \qquad & \boldsymbol{E}\{\boldsymbol{\mathcal{S}}\} \stackrel{!}{=} \mathbf{0}_n, \quad \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}\} \\ & \boldsymbol{E}\{\boldsymbol{\mathcal{N}}\} \stackrel{!}{=} \mathbf{0}_n, \quad \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{N}}\} \\ & \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{N}}\} \stackrel{!}{=} \mathbb{O}_n \end{aligned}}$$

$$\boldsymbol{\Sigma} := \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}}\}$$

Best **L**inear **U**nbiased **P**redictor — BLUP see e.g.Moritz (1980); Schuh (2016)

$$\boxed{\begin{aligned} \tilde{\boldsymbol{\mathcal{X}}} &= \left(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mathcal{L}} \\ \boldsymbol{\Sigma}\{\tilde{\boldsymbol{\mathcal{X}}}\} &= \left(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1} \end{aligned}}$$

Least Squares Collocation - BLUP

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\boxed{\begin{aligned} \boldsymbol{\mathcal{L}} = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}} \qquad & \boldsymbol{E}\{\boldsymbol{\mathcal{S}}\} \stackrel{!}{=} \boldsymbol{0}_n, \quad \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}\} \\ & \boldsymbol{E}\{\boldsymbol{\mathcal{N}}\} \stackrel{!}{=} \boldsymbol{0}_n, \quad \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{N}}\} \\ & \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{N}}\} \stackrel{!}{=} \mathbb{O}_n \end{aligned}}$$

$$\boldsymbol{\Sigma} := \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}}\}$$

Best **L**inear **U**nbiased **P**redictor — BLUP see e.g.Moritz (1980); Schuh (2016)

$$\boxed{\begin{aligned} \widetilde{\boldsymbol{\mathcal{X}}} &= \left(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mathcal{L}} \\ \boldsymbol{\Sigma}\{\widetilde{\boldsymbol{\mathcal{X}}}\} &= \left(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1} \end{aligned}}$$

reducted observations:

$$\begin{aligned} \widetilde{\Delta\boldsymbol{\mathcal{L}}} &:= \boldsymbol{\mathcal{L}} - \boldsymbol{A}\widetilde{\boldsymbol{\mathcal{X}}} \\ \boldsymbol{\Sigma}\{\widetilde{\Delta\boldsymbol{\mathcal{L}}}\} &= \boldsymbol{\Sigma} - \boldsymbol{A}\left(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T \end{aligned}$$

Least Squares Collocation - BLUP

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\boxed{\begin{aligned} \boldsymbol{\mathcal{L}} = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}} \qquad & \boldsymbol{E}\{\boldsymbol{\mathcal{S}}\} \stackrel{!}{=} \boldsymbol{0}_n, \quad \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}\} \\ & \boldsymbol{E}\{\boldsymbol{\mathcal{N}}\} \stackrel{!}{=} \boldsymbol{0}_n, \quad \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{N}}\} \\ & \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{N}}\} \stackrel{!}{=} \mathbb{O}_n \end{aligned}}$$

$$\boldsymbol{\Sigma} := \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}}\}$$

Best **L**inear **U**nbiased **P**redictor — BLUP see e.g.Moritz (1980); Schuh (2016)

$$\boxed{\begin{aligned} \widetilde{\boldsymbol{\mathcal{X}}} &= \left(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mathcal{L}} \\ \boldsymbol{\Sigma}\{\widetilde{\boldsymbol{\mathcal{X}}}\} &= \left(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1} \end{aligned}}$$

reducted observations:

$$\widetilde{\Delta\boldsymbol{\mathcal{L}}} := \boldsymbol{\mathcal{L}} - \boldsymbol{A}\widetilde{\boldsymbol{\mathcal{X}}}$$

$$\boldsymbol{\Sigma}\{\widetilde{\Delta\boldsymbol{\mathcal{L}}}\} = \boldsymbol{\Sigma} - \boldsymbol{A}\left(\boldsymbol{A}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^T$$

$$\boxed{\begin{aligned} \widetilde{\boldsymbol{\mathcal{S}}} &= \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}+\boldsymbol{\mathcal{N}}\}\boldsymbol{\Sigma}^{-1}\widetilde{\Delta\boldsymbol{\mathcal{L}}} \\ \boldsymbol{\Sigma}\{\widetilde{\boldsymbol{\mathcal{S}}}\} &= \boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}+\boldsymbol{\mathcal{N}}\}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\{\widetilde{\Delta\boldsymbol{\mathcal{L}}}\}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\{\boldsymbol{\mathcal{S}}+\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{S}}\} \end{aligned}}$$

Least Squares Collocation - BLUP

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\boxed{\begin{aligned} \mathcal{L} = A\xi + \mathcal{S} + \mathcal{N} \qquad & E\{\mathcal{S}\} \stackrel{!}{=} 0_n, \quad \Sigma\{\mathcal{S}\} \\ & E\{\mathcal{N}\} \stackrel{!}{=} 0_n, \quad \Sigma\{\mathcal{N}\} \\ & \Sigma\{\mathcal{S},\mathcal{N}\} \stackrel{!}{=} \mathbb{O}_n \end{aligned}}$$

$$\Sigma := \Sigma\{\mathcal{S} + \mathcal{N}\}$$

Best **L**inear **U**nbiased **P**redictor — BLUP see e.g.Moritz (1980); Schuh (2016)

$$\boxed{\begin{aligned} \tilde{\mathcal{X}} &= \left(A^T \Sigma^{-1} A\right)^{-1} A^T \Sigma^{-1} \mathcal{L} \\ \Sigma\{\tilde{\mathcal{X}}\} &= \left(A^T \Sigma^{-1} A\right)^{-1} \end{aligned}}$$

reducted observations:

$$\widetilde{\Delta\mathcal{L}} := \mathcal{L} - A\tilde{\mathcal{X}}$$

$$\Sigma\{\widetilde{\Delta\mathcal{L}}\} = \Sigma - A\left(A^T \Sigma^{-1} A\right)^{-1} A^T$$

$$\boxed{\begin{aligned} \tilde{\mathcal{S}} &= \Sigma\{\mathcal{S}, \mathcal{S}+\mathcal{N}\}\, \Sigma^{-1} \widetilde{\Delta\mathcal{L}} \\ \Sigma\{\tilde{\mathcal{S}}\} &= \Sigma\{\mathcal{S}, \mathcal{S}+\mathcal{N}\}\, \Sigma^{-1} \Sigma\{\widetilde{\Delta\mathcal{L}}\} \Sigma^{-1} \Sigma\{\mathcal{S}+\mathcal{N}, \mathcal{S}\} \end{aligned}}$$

Crucial points in collocation

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- **definition of the deterministic part**
- **definition of the stochastic part**
- **overcoming numerical complexity**

Crucial points in collocation

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- **definition of the deterministic part**
  - parametric models (geometrical, physical, ...)
  - non-parametric models (finite elements, splines, ...)
    continuous C^{-1}, C^0, C^1, C^2, ...
- **definition of the stochastic part**
- **overcoming numerical complexity**

Crucial points in collocation

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- **definition of the deterministic part**
  - parametric models (geometrical, physical, ...)
  - non-parametric models (finite elements, splines, ...)
    continuous C^{-1}, C^0, C^1, C^2, ...
- **definition of the stochastic part**
  - stationary — time-variable signal
  - covariance — filter approach
  - signal $\boldsymbol{\mathcal{S}}$ is a
    - function in space ... $\boldsymbol{\mathcal{S}}(\boldsymbol{x}_\mathsf{s})$
    - function in time ... $\boldsymbol{\mathcal{S}}(\boldsymbol{x}_\mathsf{t})$
    - joint spatio-temporal ... $\boldsymbol{\mathcal{S}}(\boldsymbol{x}_s, \boldsymbol{x}_t)$
    - separable ... $\boldsymbol{\mathcal{S}}(\boldsymbol{x}_\mathsf{s}) \circ \boldsymbol{\mathcal{S}}(\boldsymbol{x}_\mathsf{t})$
- **overcoming numerical complexity**

Crucial points in collocation

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- **definition of the deterministic part**
  - parametric models (geometrical, physical, ...)
  - non-parametric models (finite elements, splines, ...)
    continuous C^{-1}, C^0, C^1, C^2, ...
- **definition of the stochastic part**
  - stationary — time-variable signal
  - covariance — filter approach
  - signal $\mathcal{S}$ is a
    - function in space ... $\mathcal{S}(x_\mathrm{s})$
    - function in time ... $\mathcal{S}(x_\mathrm{t})$
    - joint spatio-temporal ... $\mathcal{S}(x_s, x_t)$
    - separable ... $\mathcal{S}(x_\mathrm{s}) \circ \mathcal{S}(x_\mathrm{t})$
- **overcoming numerical complexity**
  - data distribution (irregular - regular - gridded - equidistant)
  - infinite - finite relations (dense - sparse systems)

Numerics - Linear Equation Solver

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

	properties	algorithms
	conventional, all-purpose and trouble-free	factorization strategies Gauß, Cholesky and substitution strategies
	very robust, overcomes all obstacles	orthogonalisation strategies singular value decomposition, spectral-analysis
	rapid, opens new possibilities	sparse systems band- & profile strategies, reordering
	sometimes sovereign, often improper	iterative strategies relaxation-& gradient techniques, preconditioner
	extreme efficient, limited applications	regular structures Toeplitz- or circulating systems, discrete filter, separability

Least Squares Collocation

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\mathcal{L} = A\xi + \mathcal{S} + \mathcal{N} \qquad E\{\mathcal{S}\}, \Sigma\{\mathcal{S}\}$$

Task: **Adjustment / Regression / Data Fitting with colored signal $\mathcal{S}$**

Strategies:

- **Least Squares Collocation — filter approach**
- **Least Squares Collocation — covariance approach**

Least Squares Collocation — filter approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Least Squares Collocation (LSC) in row-wise sequential formulation

$$\boxed{\mathcal{L}_t = \boldsymbol{A}_t\boldsymbol{\xi} + \mathcal{S}_t; \quad t \in \mathbb{Z} \quad \text{with} \quad T = T_0 + t\Delta t}$$

with

- the random variable $\mathcal{L}_t$ describes the measurement process with the measurement ℓ_t as special realization,
- $\boldsymbol{A}_t$ the tth-row of the design matrix $\boldsymbol{A}$,
- $\boldsymbol{\xi}$ the unknown true paramaters (x is a realization),
- the random variable $\mathcal{S}_t$ belongs to a stochastic process $\{\boldsymbol{\mathcal{S}}_t\}_{\Delta t}$,
- the stochastic process $\{\boldsymbol{\mathcal{S}}_t\}_{\Delta t}$ can be represented by an autoregressive parameter model — AR(p) process

$$\boxed{\mathcal{S}_t = \alpha_1\mathcal{S}_{t-1} + \alpha_2\mathcal{S}_{t-2} + \ldots + \alpha_p\mathcal{S}_{t-p} + \mathcal{E}_t; \quad \boldsymbol{\mathcal{E}} \sim WN(\mathbf{0}, \boldsymbol{I}\sigma^2_{\mathcal{E}}).}$$

aim: compute $\widetilde{\boldsymbol{x}}$ as realization of the best linear unbiased estimator $\widetilde{\boldsymbol{\mathcal{X}}}$ for $\boldsymbol{\xi}$ ($\widetilde{\boldsymbol{\mathcal{X}}}$ has minimal variance $\boldsymbol{\Sigma}\{\widetilde{\boldsymbol{\mathcal{X}}}\}$).

Least Squares Collocation — filter approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Whitening strategy: Reorder AR(p)-process

$$\mathcal{S}_t = \alpha_1 \mathcal{S}_{t-1} + \alpha_2 \mathcal{S}_{t-2} + \ldots + \alpha_p \mathcal{S}_{t-p} + \mathcal{E}_t$$

with respect to the white-noise component $\mathcal{E}_t$

$$\mathcal{S}_t - \alpha_1 \mathcal{S}_{t-1} - \alpha_2 \mathcal{S}_{t-2} - \ldots - \alpha_p \mathcal{S}_{t-p} = \mathcal{E}_t \ .$$

Least Squares Collocation — filter approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Whitening strategy: Reorder AR(p)-process

$$\mathcal{S}_t = \alpha_1 \mathcal{S}_{t-1} + \alpha_2 \mathcal{S}_{t-2} + \ldots + \alpha_p \mathcal{S}_{t-p} + \mathcal{E}_t$$

with respect to the white-noise component $\mathcal{E}_t$

$$\mathcal{S}_t - \alpha_1 \mathcal{S}_{t-1} - \alpha_2 \mathcal{S}_{t-2} - \ldots - \alpha_p \mathcal{S}_{t-p} = \mathcal{E}_t \ .$$

$$\underbrace{\begin{bmatrix} \ddots & & \ddots & \ddots & \ddots & & & \\ & -\alpha_p & \ldots & -\alpha_2 & -\alpha_1 & 1 & & \\ & & -\alpha_p & \ldots & -\alpha_2 & -\alpha_1 & 1 & \\ & & & -\alpha_p & \ldots & -\alpha_2 & -\alpha_1 & 1 \\ & & & & \ddots & & \ddots & \ddots & \ddots \end{bmatrix}}_{\boldsymbol{H}} \underbrace{\begin{bmatrix} \vdots \\ \mathcal{S}_{t-2} \\ \mathcal{S}_{t-1} \\ \mathcal{S}_t \\ \vdots \end{bmatrix}}_{\boldsymbol{\mathcal{S}}} = \underbrace{\begin{bmatrix} \vdots \\ \mathcal{E}_{t-2} \\ \mathcal{E}_{t-1} \\ \mathcal{E}_t \\ \vdots \end{bmatrix}}_{\boldsymbol{\mathcal{E}}}$$

$\Longrightarrow$ **linear - time invariant - non recursive - causal - filter**

Least Squares Collocation — filter approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\boxed{\overline{\boldsymbol{\mathcal{L}}} = \overline{\boldsymbol{A}}\boldsymbol{\xi} + \boldsymbol{\mathcal{E}}; \quad \boldsymbol{\mathcal{E}} \sim WN(\boldsymbol{0}, \boldsymbol{I}\sigma_{\mathcal{E}}^2)}$$

$\Longrightarrow$ BLUE: $\overline{\boldsymbol{\mathcal{X}}} = (\overline{\boldsymbol{A}}^T\overline{\boldsymbol{A}})^{-1}\overline{\boldsymbol{A}}^T\overline{\boldsymbol{\mathcal{L}}}$ $\qquad \boldsymbol{\Sigma}\{\overline{\boldsymbol{\mathcal{X}}}\} = (\overline{\boldsymbol{A}}^T\overline{\boldsymbol{A}})^{-1}\sigma_{\mathcal{E}}^2$

$$\overline{\boldsymbol{x}} = (\overline{\boldsymbol{A}}^T\overline{\boldsymbol{A}})^{-1}\overline{\boldsymbol{A}}^T\overline{\boldsymbol{\ell}}$$

measurement vector $\boldsymbol{\ell}$ and the whole design matrix $\boldsymbol{A}$ must be filtered by the linear time invariante non recursive causal filter

Least Squares Collocation — filter approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

pros and cons: non-recursive whitening filters

- \+ flexibility in filter design
- \+ causality of the filter
- \+ cascaded filter systems
- \+ sequential filter — sequential adjustment procedures
- \+ computational complexity $\mathcal{O}\{n\}$ $(n \times p)$
- \+ efficient computation (cache-conform algorithms)
- \+ tailored for parallel computation (scalability $\mathcal{O}\{cores\}$)

Least Squares Collocation — filter approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

pros and cons: non-recursive whitening filters

+ flexibility in filter design
+ causality of the filter
+ cascaded filter systems
+ sequential filter — sequential adjustment procedures
+ computational complexity $\mathcal{O}\{n\}$ $(n \times p)$
+ efficient computation (cache-conform algorithms)
+ tailored for parallel computation (scalability $\mathcal{O}\{cores\}$)

but treating real measurement series two conflicts occur

− conflict: **infinite** filter — **finite** time series

How to start the filter process?
Filter warmup $\Longrightarrow$ loss of data!

Least Squares Collocation — filter approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

pros and cons: non-recursive whitening filters

+ flexibility in filter design
+ causality of the filter
+ cascaded filter systems
+ sequential filter — sequential adjustment procedures
+ computational complexity $\mathcal{O}\{n\}$ $(n \times p)$
+ efficient computation (cache-conform algorithms)
+ tailored for parallel computation (scalability $\mathcal{O}\{cores\}$)

but treating real measurement series two conflicts occur

− conflict: **infinite** filter — **finite** time series

How to start the filter process?
Filter warmup $\Longrightarrow$ loss of data!

− an equidistant sequence of data is assumed!

How to deal with data gaps and outliers?
How to deal with two-dimensional data?
How to deal with irregularly distributed data?

Least Squares Collocation — covariance approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

input:	$\boldsymbol{\ell},\ \boldsymbol{\Sigma}$	$\bar{\boldsymbol{\ell}} = \boldsymbol{H}\boldsymbol{\ell},\ \bar{\boldsymbol{\Sigma}} = \mathbb{1}$
model:	$\boldsymbol{\ell} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{s} + \boldsymbol{n}$	$\bar{\boldsymbol{\ell}} = \bar{\boldsymbol{A}}\boldsymbol{x} + \bar{\boldsymbol{s}} + \bar{\boldsymbol{n}}$
principle:	$(\boldsymbol{s}+\boldsymbol{s})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{s}+\boldsymbol{n})$	$(\bar{\boldsymbol{s}}+\bar{\boldsymbol{n}})^T(\bar{\boldsymbol{s}}+\bar{\boldsymbol{n}})$

$$\Sigma = R^T R$$

$$\left(R^{-1}\right)^T = H$$

$$\bar{\ell} = \left(R^{-1}\right)^T \ell$$

$$\bar{A} = \left(R^{-1}\right)^T A$$

$$\bar{\Sigma} = \left(R^{-1}\right)^T \Sigma R^{-1} = \mathbb{1}$$

“complete decorrelation”

Least Squares Collocation — covariance approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

$$\bar{\boldsymbol{\ell}} = \boldsymbol{H}\boldsymbol{\ell}$$

$$\bar{\boldsymbol{\ell}} = (\boldsymbol{R}^{-1})^T\boldsymbol{\ell}$$

$$\boldsymbol{R}^T\bar{\boldsymbol{\ell}} = \boldsymbol{\ell}$$

$$\bar{\ell}_t = \frac{1}{r_{tt}}\left(\ell_t - \sum_{i=1}^{t-1} r_{it}\,\bar{\ell}_i\right)$$

$\Longrightarrow$ **linear - time variant - recursive - causal - filter**

Least Squares Collocation — covariance approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

pros and cons: recursive decorrelation filters

+ finite filter — finite time series
+ flexible with respect to data distribution (data gaps)
+ utilization of sparse structures — finite covariance functions

Least Squares Collocation — covariance approach

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

pros and cons: recursive decorrelation filters

+ finite filter — finite time series
+ flexible with respect to data distribution (data gaps)
+ utilization of sparse structures — finite covariance functions

− recursive procedures are time-consuming
− time variant filter - attended memory $\mathcal{O}\{n^2\}$
− computational complexity $\mathcal{O}\{n^3\}$ $\frac{1}{3}(n^3 + 8n)$
− parallel implementation is very elaborately
− scalability $\mathcal{O}\{\sqrt{cores}\}$

Least Squares Collocation (LSC)— Resumé

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

LSC — filter approach:

results in: time **in**variant
non recursive
linear causal filter

"whitening filter"

- efficient computation
- (sparse) Toeplitz structure
- infinite equidistant data

LSC — covariance approach:

results in: time variant
recursive
linear causal filter

"complete decorrelation"

- computational complexity
- sparse structure
- flexible data distribution

Least Squares Collocation (LSC)— Resumé

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

LSC — filter approach:

results in: time **in**variant
non recursive
linear causal filter

"whitening filter"

- efficient computation
- (sparse) Toeplitz structure
- infinite equidistant data

rating:

very efficient
only suitable for limited use

LSC — covariance approach:

results in: time variant
recursive
linear causal filter

"complete decorrelation"

- computational complexity
- sparse structure
- flexible data distribution

rating:

flexible, robust
unhurried

Least Squares Collocation (LSC)— Resumé

Motivation
Collocation
least squares collocation
crucial points
linear equation solver
LSC filter approach
LSC cov. approach
LSC resumé
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

LSC — filter approach:

results in: time **in**variant
non recursive
linear causal filter

"whitening filter"

- efficient computation
- (sparse) Toeplitz structure
- infinite equidistant data

rating:

very efficient
only suitable for limited use

LSC — covariance approach:

results in: time variant
recursive
linear causal filter

"complete decorrelation"

- computational complexity
- sparse structure
- flexible data distribution

rating:

flexible, robust
unhurried

Can we get the best of both systems?

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Further Reflections — Tailored System

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
  Magic Square — Yule-Walker equation ✓
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
  Magic Square — Yule-Walker equation ✓
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
  Magic Square — Yule-Walker equation ✓
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
  Yes — Cholesky approach is extreme flexible ✓
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
  Magic Square — Yule-Walker equation ✓
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
  Yes — Cholesky approach is extreme flexible ✓
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
  Magic Square — Yule-Walker equation ✓
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
  Yes — Cholesky approach is extreme flexible ✓
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
  Yes — sparse and almost Toeplitz structured filter ✓
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
  Magic Square — Yule-Walker equation ✓
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
  Yes — Cholesky approach is extreme flexible ✓
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
  Yes — sparse and almost Toeplitz structured filter ✓
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Adaptive causal non-recursive whitening filter: warmup and single gap

sparse filter with almost Toeplitz structure

main part:
results direct from the AR coefficients

warmup:
computed by Cholesky recursive forward edging

data gaps:
computed by Cholesky recursive forward edging

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Adaptive causal non-recursive whitening filter: multiple gaps

sparse filter with almost Toeplitz structure

characteristics:

#data points : 120
#model : AR(7)-process
#gaps : 21
42 45
59 65 68
100
108

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
  Magic Square — Yule-Walker equation ✓
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
  Yes — Cholesky approach is extreme flexible ✓
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
  Yes — sparse and almost Toeplitz structured filter ✓
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?
  Yes — efficient adaptive whitening filters exists ✓

Resumé and further reflections

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- Link between AR-filter coefficients and covariance function
  Can we switch from one representation to the other?
  Magic Square — Yule-Walker equation ✓
- Cholesky approach revisited
  Is it possible to create non-recursive filters?
  Yes — Cholesky approach is extreme flexible ✓
- Toeplitz systems - Cholesky reduction
  Can we take advantage of the Toeplitz structure during Cholesky reduction?
  Yes — sparse and almost Toeplitz structured filter ✓
- Identification of a suitable procedure for measurement series
  Can we combine/merge Toeplitz and Cholesky procedures to handle data gaps?
  Yes — efficient adaptive whitening filters exists ✓

Resumé AR filter

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- **pro filter approach:**
  - non-recursive filter
  - Toeplitz structure
  - efficient computation

rating:

very efficient
only suitable for limited use

- **pro covariance approach:**
  - recursive filter
  - finite filter
  - flexible data distribution

rating:

flexible, robust
unhurried

Resumé AR filter

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- **pro filter approach:**
  - non-recursive filter
  - Toeplitz structure
  - efficient computation

rating:

very efficient
only suitable for limited use

- **pro covariance approach:**
  - recursive filter
  - finite filter
  - flexible data distribution

rating:

flexible, robust
unhurried

Can we get the best of both systems?

Resumé AR filter

Motivation
Collocation
Tailored Approach
Further reflections
Adaptive filter
Resumé AR filter
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary

- **pro filter approach:**
  - non-recursive filter
  - Toeplitz structure
  - efficient computation

rating:

very efficient
only suitable for limited use

- **pro covariance approach:**
  - recursive filter
  - finite filter
  - flexible data distribution

rating:

flexible, robust
unhurried

Can we get the best of both systems?

Yes, we can!

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Example:

GOCE — time-wise approach (TIM models)

Gravity field and steady-state

Ocean

Circulation

Explorer

dedicated measurements for determinig the gravity field

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

orbit determination

Quelle: ESA

attitude control

Quelle: ESA

gravity tensor

Quelle: ESA

GOCE-Mission — data characteristics

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Orbit characteristics:	altitude: inclination: circular: sampling rate: samples/revolution: total mission time: revolutions:	260 - 220 96°5 (e<0.01) 1 sec / ∼ 8 km ∼ 5.384 ∼1400 days ∼ 22.200
SST measurements:	degree/order: #coefficients:	150 22.797
SGG measurements: $\Delta V = \begin{bmatrix} \Delta V_{xx} & \Delta V_{xy} & \Delta V_{xz} \\ \Delta V_{xy} & \Delta V_{yy} & \Delta V_{yz} \\ \Delta V_{xz} & \Delta V_{yz} & \Delta V_{zz} \end{bmatrix}$	degree/order: #coefficients: #data segments: #measurements:	**300** **90.000** **88*4** **>500 Mobs**
colored noise	< 0.005 Hz 0.005 - 0.1 Hz > 0.1 Hz	$f(1/freq)$ $const.$ (± 3 mE) $f(freq)$

Processing Challenges

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

+

- **irregular data distribution**
  (measurements in the orbit)
- **huge data set and many parameters**
  (500 Mio. measurements, 90.000 parameters)
  - efficient iterative solvers
  - massively parallel solvers

- **sun synchronuous orbit $\Longrightarrow$ polar gap**
  (ill posed problem)
- **highly correlated measurements**
  (decorrelation of a huge covariance matrix)
- **SST and SGG sensor fusion**
  SST . . . **S**atellite-to-**S**atellite **T**racking
  SGG . . . **S**atellite **G**ravity **G**radiometry

gradiometer time series

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Bodenspur

V_{zz}-Messungen

residuals = EGM08 - measurements

strong correlation in the differential accelerometer measurements

GOCE Gradiometer

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

courtesy of ESA

GOCE signal - colored noise

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Correlated measurements:

- white noise in the measurement bandwidth (0.005 Hz -0.1 Hz)
- $1/f$ characteristic with individual peaks outside the bandwidth

Challenges with GOCE data:

- number of measurements (∼500.000.000)
- long correlation length (1/rev)
- time variable behavior

GOCE signal - colored noise

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Correlated measurements:

- white noise in the measurement bandwidth (0.005 Hz -0.1 Hz)
- $1/f$ characteristic with individual peaks outside the bandwidth

Challenges with GOCE data:

- number of measurements (∼500.000.000)
- long correlation length (1/rev)
- time variable behavior

Noise parametrization:

- deterministic approaches
  - nuisance variables (empirical parameters, stochastic impulses)
  - short arc approach
- stochastic approaches
  - covariance functions/collocation
  - decorrelation by filtering

GOCE signal - colored noise

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

Correlated measurements:

- white noise in the measurement bandwidth (0.005 Hz -0.1 Hz)
- $1/f$ characteristic with individual peaks outside the bandwidth

Challenges with GOCE data:

- number of measurements (∼500.000.000)
- long correlation length (1/rev)
- time variable behavior

Noise parametrization:

- deterministic approaches
  - nuisance variables (empirical parameters, stochastic impulses)
  - short arc approach
- **stochastic approaches**
  - **covariance functions/collocation**
  - **decorrelation by filtering**

Residual analysis (GOCE-RL05 data)

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

residuals:

decorrelated residuals:

Occasionally suspicuous data and data sequences:
(V_{xx}, V_{xz} and V_{zz} a few, V_{yy} a little more)

- rough outliers
- variance changes
- trend changes (mean, slope)
- non-stationary effects

How can/should this be dealt with?

- Identification of suspicious data and robust filter estimation
  $\rightarrow$ less data but higher quality data
- Design for time-varying filters around non-stationary effects.

GOCE - Schwerefeldmodelle (RL06)

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

models	period	date of publication	#obs.	$RMS_{\Delta g}$ Atlantic/Germany	RMS_N Atlantic/Germany
RL01	01/11/2009 - 11/01/2010	06/2010	6 M	3.0/ 4.1 mGal*	10.8/ 14.6 cm*
RL02	- 05/07/2010	11/2010	19 M	1.9/ 1.7 mGal*	6.8/ 6.1 cm*
RL03	- 13/04/2011	11/2011	31 M	1.3/ 1.4 mGal*	4.8/ 5.1 cm*
RL04	-19/06/2012	04/2013	278 M	0.92/0.98 mGal*	3.3/ 3.4 cm*
RL05	-20/10/2013	11/2014	440 M	0.67/0.73 mGal*	2.5/ 2.7 cm*
RL06	-20/10/2013	06/2019	440 M	0.55/0.70 mGal*	2.0/ 2.5 cm*

* durch externe Validierung mit EGM08

GOCE-TIM06 results & products

Motivation
Collocation
Tailored Approach
Example: GOCE
measurements
data characteristics
Challenges
time series
Gradiometer
residual analysis
Modelle
Example: DInSAR
Stochastic Design
Numeric Design
Summary

G O C E gravity field

geoid heights:	±100 m
accuracy:	±1.7-2.2 cm
	±0.4-0.6 mgal
resolution:	100 km
	d/o 200

homogeneous global precision

gravity field model
(90.000 parameter)

formal accuracy
(±1.7-2.2 cm)

external validation

TIM06-EGM2008: Germany
(rms: ±2.5 cm)

TIM06+EGM2008-GPS: Germany
(rms: ±1.8-2.3 cm)

- **consistent modeling of the error budget**
- **variance/covariance information available**

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

Example:

DInSAR point cloud

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

Study region: Lower-Rhine-Embayment

open-cast brown coal mines
Hambach, Garzweiler, Inden
still active

©R. Spekking

hard coal mines
Aachener and Erkelenzer coalfield
closed 1992

©Digit WDR

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

Observations: InSAR recordings of the ERS1 und ERS2 mission

DInSAR stack with 64 frames observed at irrelgular times
in the period between 1992-05-09 and 2000-12-12

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

Processing: DInSAR-SBAS with RSG (Remote-Sensing-Graz) package

spatio-temporal point cloud with surface distortions

Esch et al. (2019a,b)

Goal: Analysis, Prediction and Sensor Fusion

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

spatio-temporal point cloud (SBAS-stack):

- − huge number of data
- + identical scatterer in each time frame — 144.302 scatterer
- − irregularly distributed in space
- − clustered in urban areas
- + organized in 64 time frames
- − irregularly distributed in time

Goal: Analysis, Prediction and Sensor Fusion

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

spatio-temporal point cloud (SBAS-stack):

- − huge number of data
- + identical scatterer in each time frame — 144.302 scatterer
- − irregularly distributed in space
- − clustered in urban areas
- + organized in 64 time frames
- − irregularly distributed in time

Challenge:

design of a spatio-temporal collocation model

Crucial points in collocation

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

- **definition of the covariance function**
- **overcoming numerical complexity**

Crucial points in collocation

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

- **definition of the covariance function**

  spatio-temporal function in general: joint function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma(\text{space}, \text{time})$$

  in particular: separable function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\text{space}) \cdot \gamma_t(\text{time})$$

  $\gamma_{sp}(\text{space})$ ... covariance function in space

  $\gamma_t(\text{time})$ ... covariance function in time

- **overcoming numerical complexity**

Crucial points in collocation

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
motivation
measurements
goal
Crucial Points
Stochastic Design
Numeric Design
Summary

- **definition of the covariance function**
  spatio-temporal function in general: joint function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma(\text{space}, \text{time})$$

  in particular: separable function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\text{space}) \cdot \gamma_t(\text{time})$$

  $\gamma_{sp}(\text{space})$ ... covariance function in space
  $\gamma_t(\text{time})$ ... covariance function in time

- **overcoming numerical complexity**
  very huge variance/covariance matrix:
  (here in particular: $144302 \times 64 \sim 9$ Mio.)

  ? consider using finite covariance functions

  ? check if a Kronecker representation of the variance/covariance matrix is possible

Modeling the covariance function in space

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

Modeling the covariance function in space

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- **data characteristics:**
  - unevenly distributed in space
  - clustered in urban regions
  - non-stationary in time

original data set

- **work around:**
  - find a homogeneous data distribution for the whole area
    $\Longrightarrow$ divide the area in quadratic regions and select randomly the same number of points for each region
  - make the covariances in space independent from the time
    $\Longrightarrow$ consider the distortions only with respect to equal time differences

suggestions from: Leonhardt (2019)

Modeling the covariance function in space

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- **data characteristics:**
  - unevenly distributed in space
  - clustered in urban regions
  - non-stationary in time

thinned data set (9.457 points)

- **work around:**
  - find a homogeneous data distribution for the whole area
    $\Longrightarrow$ divide the area in quadratic regions and select randomly the same number of points for each region
  - make the covariances in space independent from the time
    $\Longrightarrow$ consider the distortions only with respect to equal time differences

suggestions from: Leonhardt (2019)

Modeling the covariance function in space

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- empirical spatial covariance functions for lag Δx

Modeling the covariance function in space

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- empirica

- adapt an analytical covariance model

$$\gamma_{sp}(\Delta x) = ae^{-b|\Delta x|}J0(\Delta x)$$

Modeling the covariance function in space

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- empirica

- adapt a

- adapt an **finite** covariance function

$$\gamma_{sp}(\Delta x) = f_{S2}(\Delta x, R)$$

Sansò and Schuh (1987); Schuh (2016)

Modeling the covariance function in space

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- er

- **or better:**
  use the new family of **flexible finite** covariance functions constructed by autocorrelation of polynomial base functions

Schubert and Schuh (2023)

Modeling the covariance function in time

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

Modeling the covariance function in time

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- **data characteristics:**
  - non stationary in time
  - unevenly distributed in time
  - time slices / frames
- **empirical temporal variances/covariances:**

variance/covariance matrix

variances

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

Modeling of the non-stationary processes with Time-Variable AutoRegressive (TVAR) processes

- **Approach:** time-variable TVAR(1) processes:

$$\mathcal{S}_t := \alpha_t \, \mathcal{S}_{t-1} + \mathcal{E}_t = \prod_{j=1}^{t} \alpha_j \, \mathcal{S}_0 + \sum_{k=1}^{t} \prod_{j=k+1}^{t} \alpha_j \, \mathcal{E}_k$$

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

Modeling of the non-stationary processes with Time-Variable AutoRegressive (TVAR) processes

- **Approach:** time-variable TVAR(1) processes:

$$\mathcal{S}_t := \alpha_t \, \mathcal{S}_{t-1} + \mathcal{E}_t = \prod_{j=1}^{t} \alpha_j \, \mathcal{S}_0 + \sum_{k=1}^{t} \prod_{j=k+1}^{t} \alpha_j \, \mathcal{E}_k$$

- Variances/covariances derived by the second central moment:

$$\begin{aligned}\gamma_t(t, \Delta t) =& \Sigma \left\{ \mathcal{S}_t, \mathcal{S}_{t+\Delta t} \right\} \\ =& E \left\{ (\mathcal{S}_t - E\{\mathcal{S}_t\})(\mathcal{S}_{t+\Delta t} - E\{\mathcal{S}_{t+\Delta t}\}) \right\} \\ =& \prod_{n=t+1}^{t+\Delta t} \alpha_n \underbrace{\left(\prod_{j=1}^{t} \alpha_j^2 \, \sigma_{\mathcal{S}_0}^2 + \sum_{k=1}^{t} \prod_{j=k+1}^{t} \alpha_j^2 \, \sigma_{\mathcal{E}}^2 \right)}_{= \gamma_t(t, 0)}, \quad \Delta t > 0\end{aligned}$$

Schuh et al. (2023); Korte et al. (2023); Korte (2024)

Modeling the covariance function in time

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- **empirical temporal variances/covariances:**

- **analytical covariance function**

Crucial points in collocation

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- **definition of the covariance function**

  spatio-temporal function in general: joint function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma(\text{space}, \text{time})$$

  in particular: separable function

  $$\gamma(\Delta\boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

  $\gamma_{sp}(\Delta x)$... homogeneous covariance function in space
  $\gamma_t(t, \Delta t)$... time-variant covariance function in time

- **overcoming numerical complexity**

Crucial points in collocation

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
CovFun Space
CovFun Time
Numeric Design
Summary

- **definition of the covariance function**

  spatio-temporal function in general: joint function

  $$\gamma(\boldsymbol{x}, \boldsymbol{t}) = \gamma(\text{space}, \text{time})$$

  in particular: separable function

  $$\gamma(\Delta\boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

  $\gamma_{sp}(\Delta x)$... homogeneous covariance function in space
  $\gamma_t(t, \Delta t)$... time-variant covariance function in time

- **overcoming numerical complexity**

  very huge variance/covariance matrix:
  (here in particular now: $9.457 \times 32 \sim 300.000$ rows/columns)

  ✓ consider using finite covariance functions

  ? check if an approach via Kronecker representation of the variance/covariance matrix is possible

Overcome numerical complexity

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Is a Kronecker representation possible?

- Covariance function is separable!

$$\gamma(\Delta x; t, \Delta t) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

Overcome numerical complexity

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Is a Kronecker representation possible?

- Covariance function is separable!

$$\gamma(\Delta x; t, \Delta t) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

- identical scatterers $\boldsymbol{x}_o$ in each time slice $\boldsymbol{t}_o$!
  $\boldsymbol{x}_o \ldots$ observed positions, $\boldsymbol{t}_o \ldots$ observed time

Overcome numerical complexity

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Is a Kronecker representation possible?

- Covariance function is separable!

$$\gamma(\Delta x; t, \Delta t) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

- identical scatterers $\boldsymbol{x}_o$ in each time slice $\boldsymbol{t}_o$!
  $\boldsymbol{x}_o \ldots$ observed positions, $\boldsymbol{t}_o \ldots$ observed time

- variance/covariance matrix of the signal is separable!
  - variance/covariance matrix between the observed points
    $\boldsymbol{\Sigma}_{\boldsymbol{S}}\{\boldsymbol{x}_o, \boldsymbol{t}_o; \boldsymbol{x}_o, \boldsymbol{t}_o\} = \boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$
  - covariance matrix between predicted and observed points
    $\boldsymbol{\Sigma}_{\boldsymbol{S}}\{\boldsymbol{x}_p, \boldsymbol{t}_p; \boldsymbol{x}_o, \boldsymbol{t}_o\} = \boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\}$
  - covariance matrix of noise
    $\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{N}}}\{\boldsymbol{x}_o, \boldsymbol{t}_o; \boldsymbol{x}_o, \boldsymbol{t}_o\} := \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{N}}}$

Prediction

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Predictor:

$$\widetilde{\boldsymbol{s}}_p = \boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \underbrace{\left(\boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \boldsymbol{\Sigma}_{\mathcal{N}} \right)}_{\text{Kronecker representation ???}}^{-1} \Delta\boldsymbol{\ell}$$

$$\boldsymbol{\Sigma}^t_{\boldsymbol{S}+\mathcal{N}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}+\mathcal{N}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$$

if a Konecker representation is possible then a **compact predictor** exists!

$$\boxed{\widetilde{\boldsymbol{S}}_p = \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \left(\boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}+\mathcal{N}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}\right)^{-1} \Delta\boldsymbol{L} \left(\boldsymbol{\Sigma}^t_{\boldsymbol{S}+\mathcal{N}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\}\right)^{-1} \boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_o, \boldsymbol{t}_p\}}$$

and the computation can be split up into a time-wise and space-wise component.

confer e.g. Blaha (1977); Rauhala (1974)

Prediction

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Predictor:

$$\widetilde{\boldsymbol{s}}_p = \boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \underbrace{\left(\boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{N}}} \right)}_{\text{Kronecker representation ???}}^{-1} \Delta\boldsymbol{\ell}$$

$$\boldsymbol{\Sigma}^t_{\boldsymbol{S}+\boldsymbol{\mathcal{N}}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}+\boldsymbol{\mathcal{N}}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$$

if a Konecker representation is possible then a **compact predictor** exists!

$$\boxed{\widetilde{\boldsymbol{S}}_p = \boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \left(\boldsymbol{\Sigma}^{sp}_{\boldsymbol{S}+\boldsymbol{\mathcal{N}}}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}\right)^{-1} \Delta\boldsymbol{L} \left(\boldsymbol{\Sigma}^t_{\boldsymbol{S}+\boldsymbol{\mathcal{N}}}\{\boldsymbol{t}_o, \boldsymbol{t}_o\}\right)^{-1} \boldsymbol{\Sigma}^t_{\boldsymbol{S}}\{\boldsymbol{t}_o, \boldsymbol{t}_p\}}$$

and the computation can be split up into a time-wise and space-wise component.

confer e.g. Blaha (1977); Rauhala (1974)

Prediction

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Predictor:

$$\widetilde{\boldsymbol{s}}_p = \boldsymbol{\Sigma}_{\boldsymbol{s}}^{t}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\boldsymbol{s}}^{sp}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \underbrace{\left(\boldsymbol{\Sigma}_{\boldsymbol{s}}^{t}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\boldsymbol{s}}^{sp}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \boldsymbol{\Sigma}_{\mathcal{N}} \right)}_{\text{Kronecker representation ???}}^{-1} \Delta\boldsymbol{\ell}$$

Prediction

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Predictor:

$$\widetilde{\boldsymbol{s}}_p = \boldsymbol{\Sigma}_{\boldsymbol{S}}^{t}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\boldsymbol{S}}^{sp}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \Bigg(\underbrace{\boldsymbol{\Sigma}_{\boldsymbol{S}}^{t}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\boldsymbol{S}}^{sp}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \boldsymbol{\Sigma}_{\mathcal{N}}}_{\text{Kronecker representation ???}} \Bigg)^{-1} \Delta\boldsymbol{\ell}$$

possible representations for the noise:

- $\boldsymbol{\Sigma}_{\mathcal{N}} = \boldsymbol{\Sigma}_{\boldsymbol{S}}^{t}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \mathbb{1}_{sp}\, \sigma_{sp}^2$ Kronecker of $(\cdot)$ exists: yes but: interpretation?
- $\boldsymbol{\Sigma}_{\mathcal{N}} = \mathbb{1}_{t}\, \sigma_{t}^2 \otimes \boldsymbol{\Sigma}_{\boldsymbol{S}}^{sp}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$ Kronecker of $(\cdot)$ exists: yes but: interpretation?
- $\boldsymbol{\Sigma}_{\mathcal{N}} = \mathbb{1}_{t} \otimes \mathbb{1}_{sp}\, \sigma_{\mathcal{N}}^2 = \mathbb{1}\sigma_{\mathcal{N}}^2$ Kronecker of $(\cdot)$ exists: no? white (i.i.d.) noise

Prediction

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Predictor:

$$\widetilde{\boldsymbol{s}}_p = \boldsymbol{\Sigma}_{\boldsymbol{s}}^{t}\{\boldsymbol{t}_p, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\boldsymbol{s}}^{sp}\{\boldsymbol{x}_p, \boldsymbol{x}_o\} \Bigg(\underbrace{\boldsymbol{\Sigma}_{\boldsymbol{s}}^{t}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \boldsymbol{\Sigma}_{\boldsymbol{s}}^{sp}\{\boldsymbol{x}_o, \boldsymbol{x}_o\} + \boldsymbol{\Sigma}_{\mathcal{N}}}_{\text{Kronecker representation ???}} \Bigg)^{-1} \Delta\boldsymbol{\ell}$$

possible representations for the noise:

- $\boldsymbol{\Sigma}_{\mathcal{N}} = \boldsymbol{\Sigma}_{\boldsymbol{s}}^{t}\{\boldsymbol{t}_o, \boldsymbol{t}_o\} \otimes \mathbb{1}_{sp}\, \sigma_{sp}^2$ Kronecker of $(\cdot)$ exists: yes but: interpretation?
- $\boldsymbol{\Sigma}_{\mathcal{N}} = \mathbb{1}_{t}\, \sigma_{t}^2 \otimes \boldsymbol{\Sigma}_{\boldsymbol{s}}^{sp}\{\boldsymbol{x}_o, \boldsymbol{x}_o\}$ Kronecker of $(\cdot)$ exists: yes but: interpretation?
- $\boldsymbol{\Sigma}_{\mathcal{N}} = \mathbb{1}_{t} \otimes \mathbb{1}_{sp}\, \sigma_{\mathcal{N}}^2 = \mathbb{1}\sigma_{\mathcal{N}}^2$ Kronecker of $(\cdot)$ exists: no? white (i.i.d.) noise

but good news for the third case:
an alternative separable approach via eigenvalue decomposition exists !

(confer Schuh et al. (2022))

Summary and Resumé

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Goal: Rigorous collocation of inhomogeneous spatio-temporal DInSAR-stack

Crucial points:

- **definition of the covariance function**
  design of separable spatio-temporal function

  $$\gamma(\Delta \boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

  ✓ $\gamma_{sp}(\Delta x)$ ... homogeneous covariance function in space
  ✓ $\gamma_t(t, \Delta t)$ ... time-variant covariance function in time
- **overcoming numerical complexity**
  very huge variance/covariance matrix:
  (here in particular now: $9.457 \times 32 \sim 300.000$ rows/columns)

  ✓ using finite covariance function
  ✓ inclusion of a realistic noise model
  ✓ rigorous solution through Kronecker representation

Summary and Resumé

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Goal: Rigorous collocation of inhomogeneous spatio-temporal DInSAR-stack

Crucial points:

- **definition of the covariance function**
  design of separable spatio-temporal function

  $$\gamma(\Delta \boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

  ✓ $\gamma_{sp}(\Delta x)$ ... homogeneous covariance function in space
  ✓ $\gamma_t(t, \Delta t)$ ... time-variant covariance function in time
- **overcoming numerical complexity**
  very huge variance/covariance matrix:
  (here in particular now: $9.457 \times 32 \sim 300.000$ rows/columns)

  ✓ using finite covariance function
  ✓ inclusion of a realistic noise model
  ✓ rigorous solution through Kronecker representation

Take home message:

Summary and Resumé

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Goal: Rigorous collocation of inhomogeneous spatio-temporal DInSAR-stack

Crucial points:

- **definition of the covariance function**
  design of separable spatio-temporal function

  $$\gamma(\Delta \boldsymbol{x}, \boldsymbol{t}) = \gamma_{sp}(\Delta x) \cdot \gamma_t(t, \Delta t)$$

  ✓ $\gamma_{sp}(\Delta x)$ ... homogeneous covariance function in space
  ✓ $\gamma_t(t, \Delta t)$ ... time-variant covariance function in time
- **overcoming numerical complexity**
  very huge variance/covariance matrix:
  (here in particular now: $9.457 \times 32 \sim 300.000$ rows/columns)

  ✓ using finite covariance function
  ✓ inclusion of a realistic noise model
  ✓ rigorous solution through Kronecker representation

Take home message:

The huge collocation problem can be rigorously solved on a notebook

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Let's have a look on the results:

continous prediction in space

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Let's have a look on the results:

continous prediction in space

equispaced sequences in time

Results of the rigorous collocation of a DInSAR-stack

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Let's have a look on the results:

continous prediction in space

equispaced sequences in time

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Let's have a look on the results:

continous prediction in space

equispaced sequences in time

Results of the rigorous collocation of a DInS

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Let's have a look on the results:

continous prediction in space

equispac

Rigorous collocation of inhomogeneous spatio-temporal signals

predictor

estimation error

Acknowledgement: This research is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Grant No. 435703911 (SCHU 2305/7-1 'Nonstationary stochastic processes in least squares collocation— NonStopLSC')

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Rigorous combination of DInSAR and levelling displacements

Next Steps

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Rigorous combination of DInSAR and levelling displacements

DInSAR displacements

levelling campaigns displacements

Next Steps

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Rigorous combination of DInSAR and levelling displacements

DInSAR displacements

levelling campaigns displacements

challenges: remove the datum definitions of DInSAR and levelling data sets

Next Steps

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Rigorous combination of DInSAR and levelling displacements

DInSAR displacements

levelling campaigns displacements

challenges:
- remove the datum definitions of DInSAR and levelling data sets
- reduce the covariances to the original content of the measurements

Next Steps

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

Rigorous combination of DInSAR and levelling displacements

DInSAR displacements

levelling campaigns displacements

challenges:
- remove the datum definitions of DInSAR and levelling data sets
- reduce the covariances to the original content of the measurements
- combine the normals of the data sets and solve the joint datum problem

Joint Adjustment of DInSAR and Levelling — First Results

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

nodes of levelling Network

Joint Adjustment of DInSAR and Levelling — First Results

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

nodes of levelling Network

point 14 (Elsdorf, Mittelstr. 31) [38m]

point 16 (Widdendorf,Horremer Str.1) [1.8km]

Joint Adjustment of DInSAR and Levelling — First Results

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Complexity
Noise Models
Results
Next steps
Summary

nodes of levelling Network

point 14 (Elsdorf, Mittelstr. 31) [38m]

m]

Variance Propagation:

	std_ges	std_lev	std_sar
$H_{14;1997}$	0.007	0.010	0.005
$H_{14;1997} - H_{16;1997}$	0.004	0.006	0.012
$H_{14;1997} - H_{16;2001}$	0.009	0.014	0.030
$(H_{14;1997.4} - H_{16;1997.4}) - (H_{14;2001.3} - H_{16;2001.3})$	0.006	0.008	0.021
$(H_{14;1998.1} - H_{16;1998.1}) - (H_{14;2000.0} - H_{16;2000.0})$	0.007	NaN	0.012

Summary and Discussion

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary
References

The goal of this lecture is to comment on the following points :

- Is it necessary to model covariances in large datasets at all?
- How to efficiently handle stochastic properties of large data sets? What methods are available?
- How and when to apply these methods?

I hope I was able to discuss some thoughts on the questions here and look forward to questions, suggestions and discussions

References

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary
References

Benoit (1924): Note Sur Une Méthode de Résolution des équations Normales Provenant de L'Application de la Méthode des Moindres Carrés a un Système D'équations Linéaires en Nombre Inférieur a Celui des Inconnues. — Application de la Méthode a la Résolution D'un Système Defini D'éQuations LinéAires. *Bulletin géodésique*, 2(1):67–77. ISSN 0007-4632. doi:10.1007/BF03031308.

Blaha, G. (1977): Least squares prediction and filtering in any dimension using the principles of array algebra. *Bulletin Géodésique*, 51:265–286. doi:10.1007/BF02525104.

Brezinski, C. (2006?): André Louis Cholesky. URL *http://www.emis.de/journals/BBMS/Bulletin/sup962/brez2.pdf*. Dedicated to Jean Meinguet on the occasion of his 65th birthday.

Brezinski, C., D. Tournès (2014): *André-Louis Cholesky - Mathematician, Topographer and Army Officer*. Birkhäuser, Springer International Publishing. ISBN-13: 9783319081342.

Brockwell, P. J., R. A. Davis (2002): *Introduction to time series and forecasting*. Springer-Verlag New York, 2nd edn.. ISBN 0-387-95351-5.

Cressie, N. A. (1991): *Statistics for Spatial Data*. Wiley.

Esch, C., J. Köhler, K. Gutjahr, W.-D. Schuh (2019a): 25 Jahre Bodenbewegung in der Niederrheinischen Bucht - Ein kombinierter Ansatz aus D-InSAR und amtlichen Leitnivellements. *Zeitschrift für Geodäsie, Geoinformation und Landmanagement*, 144(3):173–186. doi:10.12902/zfv-0257-2019. BibTeX, Pdf.

Esch, C., J. Köhler, K. Gutjahr, W.-D. Schuh (2019b): On the Analysis of the Phase Unwrapping Process in a D-InSAR Stack with Special Focus on the Estimation of a Motion Model. *Remote Sensing*, 11(19):2295. ISSN 2072-4292. doi:10.3390/rs11192295.

Isaaks, E. H., R. M. Shrivastava (1989): *Applied Geostatistics*. Oxford University Press.

Korte, J. (2024): *Non-stationary Least Squares Kollocation*. Ph.D. thesis. (in preparation).

References (cont.)

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary
References

Korte, J., T. Schubert, J. M. Brockmann, W.-D. Schuh (2023): On the estimation of time varying AR processes. In *X. Hotine-Marussi-Symposium, IAG Symposia*, Lecture Notes in Earth Science. Springer Berlin Heidelberg, Cham. doi:10.1007/1345_2023_188.

Leonhardt, J. (2019): *Raum-Zeitliche Approximation von D-InSAR-Stapeln mittels Kollokation*. Bachelorarbeit, Professur für Theoretische Geodäsie, Universität Bonn. BibTeX, PDF.

Matérn, B. (1960): Spatial Variation — Stochastic models and their application to some problems in forest survey and other sampling investigations. *Meddelanden Fran Statens Skogsforskninginstitut*, 49(5).

Moritz, H. (1980): *Advanced Physical Geodesy*. Wichmann, Karlsruhe.

Rauhala, U. (1974): *Array Algebra with applications in photogrammetry and geodesy*. Thesis, Stockholm: Royal Inst. of Techn.; Fotogrammetriska meddelanden, No. 6.

Sansò, F., W.-D. Schuh (1987): Finite covariance functions. *Bulletin Géodésique*, 61(4):331–347. doi:10.1007/BF02520559. BibTeX.

Schubert, T., W.-D. Schuh (2023): A flexible family of compactly-supported covariance functions based on cutoff polynomials. In *X. Hotine-Marussi-Symposium, IAG Symposia*, Lecture Notes in Earth Science. Springer Berlin Heidelberg, Cham. doi:10.1007/1345_2023_200.

Schuh, W.-D. (2016): Signalverarbeitung in der Physikalischen Geodäsie. In Freeden, W., R. Rummel, eds., *Handbuch der Geodäsie*, vol. Erdmessung und Satellitengeodäsie of *Springer Reference Naturwissenschaften*, pp. 73–121. Springer, Berlin Heidelberg. ISBN 978-3-662-47099-2. doi:10.1007/978-3-662-47100-5_15. BibTeX, PDF.

Schuh, W.-D., J. Korte, T. Schubert, J. M. Brockmann (2022): Modeling of inhomogeneous spatio-temporal signals by least squares collocation. In *X. Hotine-Marussi Symposium*. Milano. doi:10.48565/bonndoc-39. Oral presentation, 16.6.2022 BibTeX, PDF.

References (cont.)

Motivation
Collocation
Tailored Approach
Example: GOCE
Example: DInSAR
Stochastic Design
Numeric Design
Summary
References

Schuh, W.-D., J. Korte, T. Schubert, J. M. Brockmann (2023): Modeling of inhomogeneous spatio-temporal signals by least squares collocation. In *X. Hotine-Marussi-Symposium, IAG Symposia*, Lecture Notes in Earth Science. Springer Berlin Heidelberg, Cham. doi:10.1007/1345_2023_202.

Wackernagel, H. (2003): *Multivariate geostatistics: An Introduction with Applications*. Springer, 3rd edn..